British Physics Olympiad 2020-21

Round 2 Competition Paper

Saturday 6th February 2021

Instructions

Time: 3 hours.

Questions: All questions should be attempted.

Marks: The questions carry similar marks.

Solutions: Answers and calculations are to be written on loose paper. Students should ensure their name and school is clearly written on all answer sheets. Begin each question on a new page. Pages must be numbered. A standard formula booklet with standard physical constants may be used.

Instructions: Please do not discuss any aspect of the paper on the internet until 8 am Saturday 13th March.
Clarity: Solutions must be written legibly, in black pen, and working down the page. Scribble will not be marked and overall clarity is an important aspect of this exam paper. Diagrams should be used.

Training Dates and the International Physics Olympiad

Following this round, the best students eligible to represent the UK at the International Physics Olympiad (IPhO) will be invited to attend the **Training Camp** *to be held online his year,* (**Tuesday 6th April to Saturday 10th April 2021**). *Problem solving skills will be developed, practical skills enhanced, as well as some coverage of new material (Thermodynamics, Relativity, etc.). Five students (and possibly a reserve) will be selected for further training. From May there will be mentoring by email to cover some topics and problems.*
The IPhO this year will be held in Vilnius, Lithuania, from 17th to 25th July 2021.

Important Constants

Constant	Symbol	Value
Speed of light in free space	c	$3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Elementary charge	e	$1.60 \times 10^{-19}\,\mathrm{C}$
Acceleration of free fall at Earth's surface	g	$9.81\,\mathrm{m\,s^{-2}}$
Permittivity of free space	ε_0	$8.85 \times 10^{-12}\,\mathrm{F\,m^{-1}}$
Permeability of free space	μ_0	$4\pi \times 10^{-7}\,\mathrm{H\,m^{-1}}$
Mass of an electron	m_e	$9.11 \times 10^{-31}\,\mathrm{kg}$
Mass of a neutron	m_n	$1.67 \times 10^{-27}\,\mathrm{kg}$
Mass of a proton	m_p	$1.67 \times 10^{-27}\,\mathrm{kg}$
Radius of a nucleon	r_0	$1.2 \times 10^{-15}\,\mathrm{m}$
Planck's constant	h	$6.63 \times 10^{-34}\,\mathrm{J\,s}$
Gravitational constant	G	$6.67 \times 10^{-11}\,\mathrm{m^{3}\,kg^{-1}\,s^{-2}}$
Boltzmann constant	k	$1.38 \times 10^{-23}\,\mathrm{J\,K^{-1}}$
Molar gas constant	R	$8.31\,\mathrm{J\,mol^{-1}\,K^{-1}}$
Specific heat capacity of water	c_w	$4.19 \times 10^{3}\,\mathrm{J\,kg^{-1}\,K^{-1}}$
Mass of the Sun	M_S	$1.99 \times 10^{30}\,\mathrm{kg}$
Mass of the Earth	M_E	$5.97 \times 10^{24}\,\mathrm{kg}$
Radius of the Earth	R_E	$6.38 \times 10^{6}\,\mathrm{m}$

Qu 1. General Questions

(a) The bottle below is filled to the brim with salad dressing made from oil and vinegar. The bottle is shaken before being left to stand, whereupon it is observed that the oil slowly separates and rises to the top. Is the pressure on the bottom of the bottle now the same, greater or less than before? Explain your reasoning carefully.

Figure 1: Credit IndiaMART Glass 200ml Milk Bottle

[5]

(b) Table 1 shows the timetable of a passenger boat on the River Rhine.

Time	Town	Time	Town	Position along the river /km
1700	St Goarhausen	1851	St Goarhausen	642
1830	Bacharach	1801	Bacharach	659

Table 1: Rhine cruise timetable

Obtain a numerical estimate of the speed of the Rhine current and the speed of the boats in still water. N.B. the position markers follow the river Rhine, indicating the distance along its course from a zero marker. **[5]**

(c) Estimate the work done in inflating a bicycle tyre. Provide your own approximate data and work in SI units. **[5]**

(d) A sodium salt is heated in a flame at a temperature of 1500 K and the sodium line of wavelength 590 nm is observed. Calculate the spread of wavelengths assuming that all the molecules move with the same speed.
Mass number of sodium is 23. **[5]**

(e) A thick walled flask has an outer surface which is a sphere of radius 10 cm and an inner surface which is also spherical and concentric with the outer. When it is filled with an opaque liquid and held up against a diffuse source of light, the flask looks black all over. Is the flask able to contain 1 litre of liquid?
1 litre = 1000 cm^3 **[5]**

[25 marks]

Qu 2. Proton Decay

Proton decay is a hypothetical form of particle decay in which the proton decays into lighter subatomic particles. One possible decay mode, common to many models (including Grand unified Theories and String Theories), is into a positron and a neutral pion: $p \to e^+ + \pi^0$. The masses of these particles, expressed in units of MeV/c^2 where c is the speed of light in vacuo, are: $m_p = 938\,\mathrm{MeV}/c^2$, $m_e = 0.511\,\mathrm{MeV}/c^2$ and $m_\pi = 135\,\mathrm{MeV}/c^2$. As with radioactive decay the rate of decay is presumed to be proportional to the number of undecayed protons in a given sample.

(a) Show that $N = N_0 e^{-\lambda t}$, where N is the number of undecayed protons at time t, N_0 is the number at time $t = 0$ and λ is a constant called the decay constant. **[2]**

(b) By considering the number of protons, $\mathrm{d}N$, that decay in time $\mathrm{d}t$:

(i) Show that the probability, $\mathrm{d}P$ for a proton to decay in this time can be expressed as $\mathrm{d}P = \lambda e^{-\lambda t}\,\mathrm{d}t$. **[2]**

(ii) Show that all the probabilities (from $t = 0$ to $t = \infty$) sum to 1 as expected. **[1]**

(iii) Show that the probability that a proton remains *undecayed* after time t is given by $p(t) = e^{-\lambda t}$. **[2]**

(iv) Show that the mean time taken for a proton to decay, τ, known as the mean lifetime of the proton, is given by $\tau = 1/\lambda$.

Hint: The mean of a quantity, $f(t)$, is given by

$$\langle f(t) \rangle = \frac{\int_0^\infty f(t)p(t)\,\mathrm{d}t}{\int_0^\infty p(t)\,\mathrm{d}t}$$

You may assume, without proof, the standard integral

$$\int_0^\infty x e^{-x}\,\mathrm{d}x = 1$$

[2]

(c) The average heat outflux through the surface of the Earth is $Q = 92\,\mathrm{mW\,m^{-2}}$.

(i) Assuming that this outflux is produced entirely by proton decay, estimate a lower bound for the proton's mean lifetime. **[4]**

(ii) Why is this a considerable underestimate? **[1]**

(d) The Super-Kamiokande detector in Japan is currently the largest detector for the observation of proton decay. It consists of a tank containing $50,000\,\mathrm{tons}$ of ultrapure water with about 13,000 photomultiplier tubes, covering about 40% of the area around the tank that detect Cherenkov radiation produced by the movement of charged particles through the water.

(i) Assuming the detectors are 100% efficient, how long would you have to wait, on average, to observe *one* proton decay. **[4]**

(ii) How many decays would you expect to observe per year, on average? **[2]**

(iii) The detection process for such decays can be modelled using Poisson statistics where the probability of observing k decays per year, $p(k)$, is given by

$$p(k) = \frac{\mu^k e^{-\mu}}{k!}$$

with μ the average number of decays per year. So far, no convincing proton decays have been observed. What would be the probability of not observing any decays in a year and what would this mean for the lifetime you calculated in **(c)** **(i)**? **[3]**

(e) Assuming that any proton decay occurs via the process $p \to e^+ + \pi^0$, with the proton decaying while at rest, determine the energies and speeds of the positron and neutral pion produced. *Hint: The energy, E, of a particle of rest mass m and momentum p (velocity v) is given by $E^2 = p^2c^2 + m^2c^4$ with $p = \gamma m v$ and $\gamma = 1/\sqrt{1 - \frac{v^2}{c^2}}$.* **[3]**

[26 marks]

Qu 3. Orbits

This question is about the mechanics of circular and elliptical orbits. Consider a satellite of mass m in orbit around a planet of mass M such that it is in free fall and sufficiently far from the planet to be able to ignore atmospheric effects. The point of closest approach (perigee) is at a distance r_p from the centre of the planet and the highest point in the orbit (apogee) is at a distance r_a from the centre of the planet.

(a) As the orbit is not circular, the equation $F = mv^2/r$ does not apply at any point in the orbit. However, at the apogee and perigee the velocity vector is perpendicular to the gravitational force. Hence the angular momentum ($= mr\omega$ where ω is the angular velocity) is constant and at those two points is simply equal to $mv_\mathrm{a}r_\mathrm{a} = mv_\mathrm{b}r_\mathrm{b}$.

Use this with the conservation of energy (kinetic + gravitational potential) to derive Newton's *vis viva* equation (eq. 1) to find the velocity at any given radius of orbit on the ellipse:

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right) \qquad \text{(eq. 1)}$$

where $a = (r_\mathrm{a}+r_\mathrm{p})/2$, the semi-major axis of the ellipse, $r_\mathrm{a} = a(1+e)$ and $r_\mathrm{p} = a(1-e)$, where e is the eccentricity. Note: the largest diameter of an ellipse has a length $2a$ so that a is the length of the semi major axis. **[4]**

(b) In order to escape from orbit, the satellite undergoes a single short burn at a specific point in the orbit which gives it a new orbit with infinite semi-major axis. That burn creates a velocity change Δv which is a figure of interest to engineers as it is essentially proportional to the amount of fuel burnt in a given manoeuvre.

(i) Find expressions for the necessary Δv to reach escape velocity from apogee (Δv_a) and perigee (Δv_p). **[5]**

(ii) Hence show that the ratio:

$$\frac{\Delta v_\mathrm{a}}{\Delta v_\mathrm{p}} = \frac{\sqrt{2(1-e)} - (1-e)}{\sqrt{2(1+e)} - (1+e)}$$

[2]

(iii) Recalling that, for an ellipse, $0 < e < 1$, show that it requires a greater velocity change to launch a satellite into escape orbit from apogee than from perigee - this may be counter-intuitive! **[2]**

In order to change from one circular orbital radius (r_1) to another (r_2) in the same orbital plane, a very efficient way is to use the two-burn Hohmann transfer. A brief burn is applied at one point in the orbit which makes the orbit elliptical. The first burn point is the perigee of the ellipse. On reaching the apogee, a second burn is applied to give the necessary velocity to make a circular orbit at the new height.

(c) Show that the velocity changes at perigee and apogee are given by:

$$\Delta v_{\rm p} = \sqrt{\frac{GM}{r_1}}\left(\sqrt{\frac{2r_2}{r_1 + r_2}} - 1\right)$$

and

$$\Delta v_{\rm p} = \sqrt{\frac{GM}{r_2}}\left(1 - \sqrt{\frac{2r_1}{r_1 + r_2}}\right)$$

[5]

(d) For a given initial orbit r_1, the total $\Delta v = \Delta v_{\rm a} + \Delta v_{\rm p}$ will vary with r_2.

By differentiation with respect to r_2, keeping r_1 fixed, show that the maximum value of Δv occurs for the positive root of the equation

$$x^3 - 15x^2 - 9x - 1 = 0$$

Where $x = r_2/r_1$. This root has the approximate value of 15.6.

In other words, it requires more energy to place the satellite in a higher orbit than it does to remove it to escape velocity, another counter-intuitive result! **[4]**

[22 marks]

Qu 4. EM Waves

A travelling electromagnetic wave consists of oscillating electric and magnetic field components at right angles to each other. For the purposes of this question we shall consider just the electric part which may be written,

$$E_z = E_{z0} \sin 2\pi \left(\frac{x}{\lambda} - ft \right) \qquad \text{(eq. 1)}$$

for a wave travelling in one dimension towards the positive x direction, with wavelength λ and frequency f:
Here the electric field is assumed to be in the z direction i.e. the wave is plane-polarised.
To simplify the algebra we shall use the wave number, $k = 2\pi/\lambda$, and angular frequency, $\omega = 2\pi f$ allowing us to rewrite (eq. 1) as:

$$E_z = E_{z0} \sin (kx - \omega t) \qquad \text{(eq. 2)}$$

(a) Considering the variables x and t explain how this equation describes a travelling (progressive) wave.

By considering the substitutions $x \to x + \Delta x$ and $t \to t + \Delta t$, show how the direction and speed of propagation of the wave is determined from (eq. 2). **[3]**

(b) Write down the equation for an identical wave travelling in the opposite direction and show that the superposition of the two creates a standing wave of the form:

$$E_z^{'} = E_{z0}^{'} \sin kx \cos \omega t$$

State the value of $E_{z0}^{'}$ **[3]**

(c) A standing wave can be considered as a series of oscillations of frequency ω whose amplitude varies with position ($E_{z0}^{'} \sin kx$). Find the distance between points of zero constant displacement ("nodes") in terms of k and hence in terms of λ. In the late 1880s Heinrich Hertz used a radio frequency generator at 60 MHz and a spark detector to measure standing waves created by reflection of the radio waves from a large metal plate.

(i) Find the spacing of the nodes in his experiment. **[1]**

By calculating the frequency of the waves from the electrical circuit used to generate them and the measured wavelength, Hertz was able to calculate the speed of the waves and show it was consistent with Maxwell's analysis and that radio waves were electromagnetic and travelled at the same speed as light.

(ii) Hertz's waves did not travel in one direction but instead spread out. How would this affect the standing wave pattern produced by a reflection? **[1]**

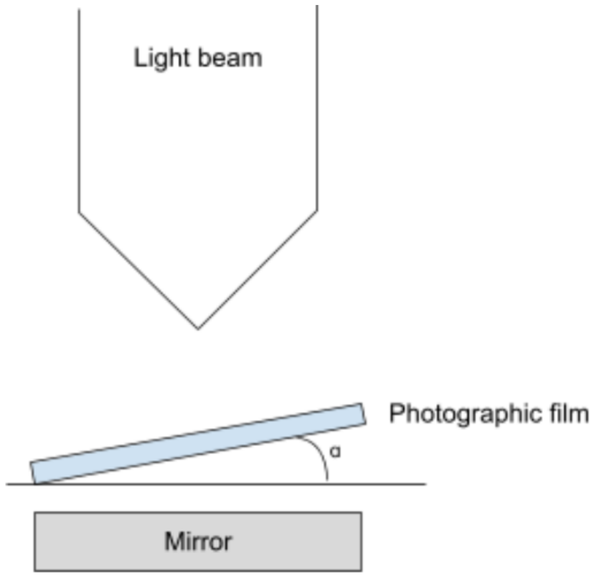

Figure 2

(d) In 1890, Otto Wiener performed a similar experiment but with light. Light of wavelength close to 400 nm was incident on a mirror, creating a standing wave. A piece of photographic film was placed at a small angle, α, (approx $2°$) across the path of the light as in **Figure 2**:

In order for Wiener to observe *fringes* (bands of light and dark) across his film he had to ensure that the light was monochromatic, the photosensitive film was much thinner than one wavelength and the the photographic film material was not reflective.

(i) Explain briefly the relevance of these points. **[3]**

(ii) Find an expression for the spacing of the fringes (dark to dark) in terms of the wavelength λ and angle α, and estimate the value for the experimental conditions given. (Hint: $2°$ is approximately 1/30 radian) **[2]**

(e) Now consider two waves of the same amplitude but with slightly different angular frequencies, ω_1 and ω_2 .

(i) Show that, where the waves superpose at some fixed position x where the two oscillations can be written as

$$E_z^1 = E_{z0} \cos \omega_1 t \qquad \text{and} \qquad E_z^2 = E_{z0} \cos \omega_2 t$$

then the resultant wave can be written in the form: $E_z'' = E_{z0}'' \cos \omega_a t \cos \omega_b t$

[1]

(ii) Write down the values of E_{z0}'' , ω_a and ω_b **[1]**

(iii) This phenomenon is known as “beats”. If two light waves whose frequencies differed by 10Hz were to superpose in this way on a screen what would be observed? **[1]**

(iv) In practice the difference in two visible light frequencies might be approximately 1 MHz. If two light sources 1 MHz apart in frequency (f, not ω) are incident on a fast photodiode and the output displayed on an oscilloscope with timebase set to $0.2\,\mu s$ (i.e. 1 small square on the oscilloscope corresponds to $0.2\,\mu s$), sketch what would be seen, assuming the display is 10 squares wide. **[2]**

(v) A researcher wishing to measure the frequency difference can measure on the oscilloscope to a precision of 1/20 of a square using digital measuring tools built into the device. Calculate the percentage uncertainty in his measurement of the frequency difference. **[2]**

(vi) Estimate a typical visible light frequency in the green part of the spectrum. To how many significant figures would the two frequencies have to be measured in order to achieve the same uncertainty in the difference between them? **[2]**

It should, thus, be evident that a direct measurement of differences through interference is highly sensitive compared to measuring frequencies directly.

(f) Now consider the typical Young's slits arrangement. Rather than following the typical analysis, consider it to be like Wiener's experiment. The two light waves, rather than being intercepted at an angle by a film, meet at angle on a screen as in **Figure 3** and produce fringes separated by a distance y.

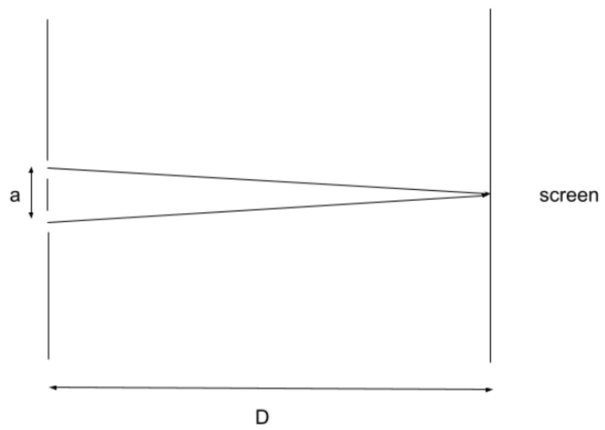

Figure 3

(i) By considering the two beams with their wavefronts to be coming from the two slits at an angle to each other, and the wavefronts meeting at the screen (see **Figure 4**), find the equivalent of α in terms of a and D and write an expression for y. Compare it to the standard Young's slits expression. **[2]**

(ii) Does the analogy hold perfectly? Consider the line along the middle of the slits - this is always a bright point in Young's slits (i.e. where the path difference between the two beams is zero). In Wiener's experiment, when the path difference is zero (where the photographic film just touches the mirror) is a dark point. Explain. **[1]**

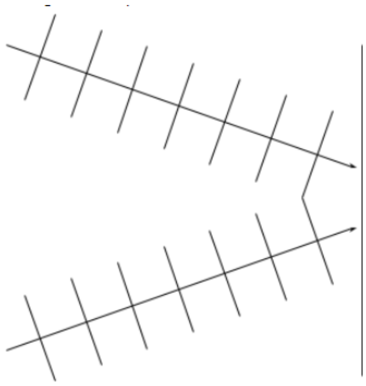

Figure 4

(iii) Away from the central fringe, any other bright fringe will occur where the path difference between the two rays is equal to an integer number of wavelengths. Show that the fringes lie along a hyperbola for $D \gg a$. Why is it not true for smaller values of D? **[2]**

[27 marks]

END OF PAPER