Question 1

a) This can be done in a descriptive manner. A quantitative calculation is not asked for in the question.
The key points to look for are

- the bubbles reach a terminal velocity as they rise, so that there is no resultant force.
- the weight of the gas can be neglected
- the upthrust is proportional to the volume, $U \propto R^3$
- The drag non the bubble is proportional to the area (R^2) and $\propto v$ or v^2
- and hence v or $v^2 \propto R$ (so larger bubble rise faster)

An air bubble experiences three key forces:

- Weight $W = \rho_a V g$
- Upthrust $U = \rho V g$ (Archimedes' principle)
- Drag $D = \frac{1}{2} C_D \rho A v^2$ (assuming inertial drag dominates over viscous drag)

Since density of air $\rho_a << \rho$, the density of fluid, weight negligible.

At terminal velocity $U = D \Rightarrow \rho V g = \frac{1}{2} C_D \rho A v^2$

but $V \propto R^3$, $A \propto R^2$ $\Big\} \Rightarrow v \propto \sqrt{R}$ $\Rightarrow$ larger bubbles rise more quickly because upthrust scales more rapidly with size than does drag.

Note that all bubbles will expand as they rise due to decreasing ambient pressure with height.
This observation does not affect main conclusion above.

[5]

b) In these explanations, the observations that are being explained should be included.

(i) Crystals have grown on the nail. As water vapour from the damp air settles in the morning, it releases a latent heat. this is not easily conducted away by the wood, but the nail will conduct it away and so the crystals grow on the nail. **[3]**

(ii) An example of crystal growth. A lot of straight lines with branches coming out. the observation gains one or two marks, But not "theer is ice on the windscreen". **[3]**

c) 200 (peas) × 52 × 2 (twice a week) × $(60 \times 10^6)/3$ (a third of the population) = 4×10^{11}
An answer between 5×10^{10} and 10^{12} **[5]**

d) (i) i. A sketch showing the **same** current flowing from the left of R_{in} and through R_{f} **[51**

ii.

$$V_{\text{in}} + \frac{V_{\text{out}}}{A_0} = iR_{\text{in}}$$

$$-\frac{V_{\text{out}}}{A_0} - V_{\text{out}} = iR_{\text{f}}$$

[2]

iii. Eliminating i

$$\frac{V_{\text{in}}}{R_{\text{in}}} + \frac{V_{\text{out}}}{R_{\text{in}}A_0} = -\frac{V_{\text{out}}}{R_{\text{f}}A_0} - \frac{V_{\text{out}}}{R_{\text{f}}}$$

$$V_{\text{out}}\left(\frac{1}{R_{\text{f}}} + \frac{1}{R_{\text{f}}A_0} + \frac{1}{A_0R_{\text{in}}}\right) = -\frac{V_{\text{in}}}{R_{\text{in}}}$$

For $A \approx 10^5$,

$$\frac{V_{\text{out}}}{V_{\text{in}}} = A_{\text{g}} = -\frac{R_{\text{f}}}{R_{\text{in}}}$$

[2]

(ii) The non-inverting amplifier.
In the UK paper, there was a typo and it was stated

$$v = \frac{V_{\text{out}}}{A_0}$$

with V_{in} missing, instead of the correct

$$V_{\text{out}} = A_0(V_{\text{in}} - v)$$

They can obtain the current i and relate v and V_{out}.
But they cannot obtain the gain as V_{in} is missing. So you will just have to make a judgement call as to whether they are on the right track and should be given full marks for getting close. This is allowable. There are only one or two marks for getting the extra final part, and they may bring in V_{in} themselves.

In the China paper, the correction was made: $V_{\text{out}} = A_0(V_{\text{in}} - v)$

i.

$$i = \frac{V_{\text{out}}}{R_1 + R_2} \text{ and } i = \frac{v}{R_2}$$

[2]

ii. Eliminate i,

$$v = \frac{R_2 V_{\text{out}}}{R_1 + R_2}$$

Given that

$$V_{\text{out}} = A_0 (V_{\text{in}} - v)$$

then

$$V_{\text{out}} = A_0 \left(V_{\text{in}} - \frac{R_2 V_{\text{out}}}{R_1 + R_2} \right)$$

$$V_{\text{out}} \left(1 + \frac{R_2 A_0}{R_1 + R_2} \right) = A_0 V_{\text{in}}$$

$$V_{\text{out}} \left(\frac{1}{A_0} + \frac{R_2}{R_1 + R_2} \right) = V_{\text{in}}$$

So using $A_0 \gg 1$

$$\frac{V_{\text{out}}}{V_{\text{in}}} = A_{\text{g}} \approx \left(1 + \frac{R_1}{R_2} \right)$$

[2]

(iii) i.

$$i = \frac{V_{\text{in}}}{R_{\text{in}}}$$

$$0 - V_{\text{out}} = \frac{Q}{C}$$

[2]

ii.

$$-\frac{dV_{\text{out}}}{dt} = \frac{1}{C} \frac{dQ}{dt}$$

so that

$$-\frac{dV_{\text{out}}}{dt} = \frac{i}{C}$$

Therefore

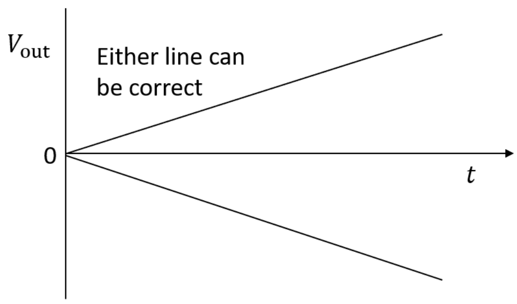
$$-\frac{dV_{\text{out}}}{dt} = \frac{V_{\text{in}}}{R_{\text{in}} C}$$

If V_{in} is constant,

$$\int_0^{V_{\text{out}}} dV_{\text{out}} = -\frac{V_{\text{in}}}{R_{\text{in}} C} \int_0^t dt$$

$$V_{\text{out}} = -\frac{V_{\text{in}}}{R_{\text{in}} C} t$$

Sketch gra



[3]

Question 2 Energy Levels

a) (i) $n\lambda = 2\pi r$ or $2n.\frac{\lambda}{2}$ = circumference **[1]**

(ii) Newton II

$$\frac{mv^2}{r} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2}$$

energy of system

$$E = \frac{1}{2}mv^2 - \frac{1}{4\pi\epsilon_0}\frac{e^2}{r}$$
$$= -\frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{e^2}{r}$$

[3]

(iii) Using

$$n\lambda = 2\pi r$$
$$n\frac{h}{p} = 2\pi r$$

so that

$$n\frac{h}{2\pi} = rp$$

Or $n\hbar = rp$ with the notation $\hbar = \frac{h}{2\pi}$ **[2]**

(iv) Using

$$\frac{mv^2}{r} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2}$$
$$\frac{p^2 r}{m} = \frac{1}{4\pi\epsilon_0}e^2$$

and from $n\hbar = rp$ we can rearrange to form $\frac{n^2\hbar^2}{r} = p^2 r$

So that

$$\frac{n^2\hbar^2}{mr} = \frac{1}{4\pi\epsilon_0}e^2$$

So

$$r = \frac{4\pi\epsilon_0 n^2\hbar^2}{me^2}$$

and for $n = 1$ we can calculate r

$$r = \frac{\hbar^2}{m\frac{1}{4\pi\epsilon_0}e^2} = 0.53 \times 10^{-10}\,\text{m}$$

$$r = 5.3\,\text{nm}$$

This is a suitable value. No mark for a comment of this sort - it is to make sure you look at your answer. **[3]**

(v)

$$E = -\frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{e^2}{r}$$

and substituting for r

$$E = -\frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{me^4}{\hbar^2 n^2}$$

But we could also take a direct approach to this $n = 1$ energy level

$$p = \frac{h}{\lambda}$$

$$E = -\frac{p^2}{2m} = -\frac{h^2}{\lambda^2 2m}$$

and with $n = 1$ we write $\lambda = 2\pi r$

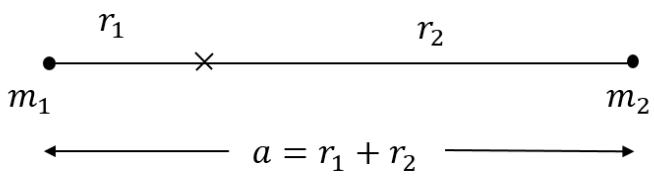
$$E = -\frac{h^2}{4\pi^2 r^2 2m}$$

$$E = -\frac{\hbar^2}{2mr^2}$$

$$E = -13.6\,\text{eV}$$

[3]

b) (i) Something like this:



An isolated system, so about the CM,

$$m_1 r_1 = m_2 r_2$$

and the centripetal force on each is due to electrostatic attraction between **p** and **e**, so

$$m_1 r_1 \omega^2 = \frac{ke^2}{a^2}$$

$$m_2 r_2 \omega^2 = \frac{ke^2}{a^2}$$

which implies the same value of ω. **[2]**

Using, $m_1 r_1 = m_2 r_2$

$$\frac{m_1}{m_2}(a - r_2) = r_2$$

$$\frac{m_1}{m_2}a = r_2\left(1 + \frac{m_1}{m_2}\right)$$

Hence,

$$a = r_2\left(\frac{m_2 + m_1}{m_1}\right)$$

So,

$$a = \frac{r_2 m_2}{\mu}$$

As a result,

$$\omega^2 = \frac{1}{4\pi\epsilon_0}\frac{e^2}{a^3}\frac{m_1}{m_2}\mu$$

[2]

(ii) since we have the angular momentum as rp, that is $mr^2\omega$. So

$$n\hbar = (m_1 r_1^2 + m_2 r^2)\omega$$
$$= (m_2 r_2)(r_1 + r_2)\omega = m_1 r_1 a\omega$$

Then

$$n\hbar = a\mu \times a\omega = a^2\mu\omega = \mu a^2\omega$$

As a result

$$(n\hbar)^2 = \frac{1}{4\pi\epsilon_0}\mu a e^2$$

which is the same form seen earlier with m replaced by μ and r replaced by a Hence

$$E = -\frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{\mu e^4}{\hbar^2 n^2}$$

For the hydrogen atom, $\mu \approx m_\mathrm{e}$ so that the $n = 1$ level remains at $-13.6\,\mathrm{eV}$ and the $n = 2$ level is $-3.4\,\mathrm{eV}$.
For tritium the energy levels differ by only about 0.02%,

[2]

since (this detail is not required)

$$\begin{aligned}\frac{m_\mathrm{e} - \mu}{m_\mathrm{e}} &= 1 - \frac{\mu}{m_\mathrm{e}} \\ &= 1 - \frac{m_\mathrm{e} 3m_\mathrm{p}}{m_\mathrm{e}(m_\mathrm{e} + 3m_\mathrm{p})} \\ &= 1 - \frac{m_\mathrm{e} 3m_\mathrm{p}}{(m_\mathrm{e} + 3m_\mathrm{p})} \\ &= 1 - \frac{\cancel{3m_\mathrm{p}}}{\cancel{3m_\mathrm{p}}\left(1 + \frac{m_\mathrm{e}}{3m_\mathrm{p}}\right)} \\ &\approx 1 - \left(1 - \frac{m_\mathrm{e}}{3m_\mathrm{p}}\right) \\ &= \frac{m_\mathrm{e}}{3m_\mathrm{p}} \approx \frac{1}{6000}\end{aligned}$$

(iii) Since $\mu < m_\mathrm{e}$ then the energy levels are of a smaller magnitude,

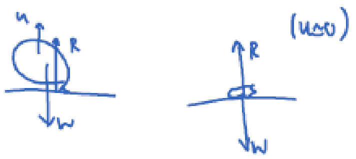
$$\begin{aligned}\mu &= \frac{m_\mathrm{e} 3m_\mathrm{p}}{m_\mathrm{e} + 3m_\mathrm{p}} \\ &= \frac{m_\mathrm{e}\cancel{3m_\mathrm{p}}}{\cancel{3m_\mathrm{p}}\left(1 + \frac{m_\mathrm{e}}{3m_\mathrm{p}}\right)} \\ &\approx m_\mathrm{e}\left(1 - \frac{m_\mathrm{e}}{3m_\mathrm{p}}\right)\end{aligned}$$

and so they are closer together for tritium and hence a lower frequency photon will cause a transition and thus a longer wavelength for tritium. Conversely, hydrogen would absorb a shorter wavelength. Comment required, not a guess.

[2]

Question 3 Balloons

a) Balloons



(b) Require, at least,

$$\underbrace{mg}_{\text{balloon weight}} + \underbrace{W_{He}}_{\text{weight of added helium}} = \underbrace{W_{air}}_{\text{weight of displaced air}} \quad \Rightarrow \quad mg = (\rho_{air} - \rho_{He})Vg$$

$$V = \frac{m}{\rho_{air} - \rho_{He}}$$

But density of ideal gas is

$$\rho = \frac{M_r P}{RT}$$

(M_r: molar mass; P: pressure; R: molar gas const.; T: abs. temp.)

$$\Rightarrow \quad V = \frac{mRT}{P(M_r^{air} - M_r^{He})} \simeq \frac{10^{-2} \times 8.31 \times 293}{10^5(29\times10^{-3} - 4\times10^{-3})} \simeq 1.0\times10^{-2}\ m^3$$

(c) For an ideal gas: $\frac{1}{2}m\langle v^2\rangle = \frac{3}{2}kT$

$$\Rightarrow \langle p^2\rangle = 3mkT, \quad p_{rms} = \sqrt{\langle p^2\rangle} = \sqrt{3mkT}$$

Direction of gas particle is random, so

$$\langle p_x^2\rangle = \langle p_y^2\rangle = \langle p_z^2\rangle \quad \text{but} \quad \langle p^2\rangle = \langle p_x^2 + p_y^2 + p_z^2\rangle = 3p^2$$

$$\Rightarrow \text{so} \quad p^2 = \frac{1}{3}p_{rms}^2 = mkT,$$

$$p = \sqrt{mkT} \quad \square$$

(d) From the definition of pressure,

$$\underbrace{PA}_{\text{Force on hole area from bombarding particles}} = \underbrace{-p\frac{dN}{dt}}_{\text{rate of arrival of momentum onto hole area per unit time}}$$

$$\Rightarrow \frac{dN}{dt} = -\frac{PA}{\sqrt{mkT}} \quad \square$$

[]

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{pV}{kT}\right) = \frac{-pA}{\sqrt{mkT}}$$

As hole small, the balloon in quasi-static equilibrium with atmosphere, so p & T are ~ constant throughout deflation:

$$\frac{p}{kT}\frac{dV}{dt} = -\frac{pA}{\sqrt{mkT}}$$

leaving $\frac{dV}{dt} = -A\sqrt{\frac{kT}{m}}$ □

(f) In 1 hour, $V = 0.95 \times 2.0 \times 10^{-2} = 1.9 \times 10^{-2}\,m^3 \Rightarrow \Delta V = -1.0 \times 10^{-3}\,m^3$

$$\Rightarrow \quad \frac{-10^{-3}}{3600} = -A\sqrt{\frac{8.31 \times 293}{4 \times 10^{-3}}}$$

(293 → estimate room temp.; 4×10^{-3} → molar mass of He-4; 8.31 → gas constant)

$$\therefore \quad A \approx 1 \times 10^{-9}\,m^2$$

But, surface area of balloon (initially) is $4\pi\left[\frac{3}{4\pi}V\right]^{2/3} = 0.36\,m^2$

so $A_{hole} \ll A_{balloon}$, as it must be for analysis above to hold.

(g) If hole significantly bigger we do not have quasi-static equilibrium. We can longer assume T, p uniform throughout balloon nor that the equal atmospheric quantities. would have to account for the flow that would develop → fluid dynamical problem!

(h) Now V is constant but p will vary over time. We will still assume effusion is slow enough that process is isothermal. Take eq^n 10; then use IGL:

$$\frac{d}{dt}\left(\frac{pV}{kT}\right) = -\frac{pa}{\sqrt{mkT}} \quad \Rightarrow \quad \frac{V}{kT}\frac{dp}{dt} = -\frac{pa}{\sqrt{mkT}}$$

so $\frac{1}{p}\frac{dp}{dt} = -\frac{a}{V}\sqrt{\frac{kT}{m}} \quad \Rightarrow \quad p = p_0 \exp\left[-\frac{a}{V}\sqrt{\frac{kT}{m}}\,t\right]$

So pressure decays exponentially as particles are lost through hole.

[since $\frac{dN}{dt} \propto -N$]