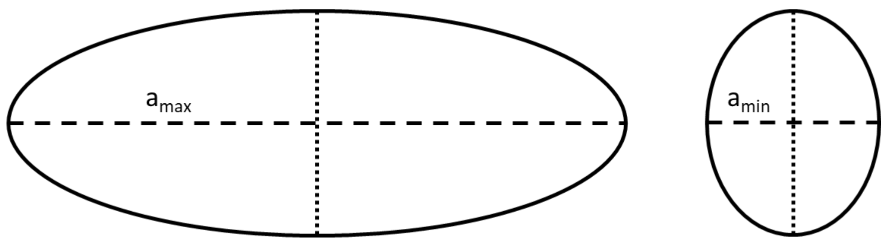
Astronomy & Astrophysics Challenge

September - December 2018

Solutions and marking guidelines

- The total mark for each question is in **bold** on the right-hand side of the table. The breakdown of the mark is below it.
- There is an explanation for each correct answer for the multiple-choice questions. However, the students are only required to write the letter corresponding to the right answer.
- In Section C, students should attempt **either** Qu 13 **or** Qu 14. If both are attempted, consider the question with the higher mark.
- Answers to two or three significant figures are generally acceptable. The solution may give more to make the calculation clear.
- There are multiple ways to solve some of the questions; please accept all good solutions that arrive at the correct answer.

Question	Answer	Mark
Section A		**10**
1.	B The velocity is tangential to the motion and the resultant force must be centripetal (arrow 2). As B is the only option with the correct force we don't need to know if the motion is clockwise/anticlockwise.	1
2.	D As the surface of a comet heats up upon getting closer to the Sun the ice sublimes and as the water vapour violently erupts from the surface it takes lots of dust with it, forming the tail.	1
3.	B The dust left by a comet's tail stays in roughly the same place over short timescales, hence why meteor showers happen at the same time every year.	1
4.	D Deneb and Vega are high in the sky at this time in the evening (part of the Summer Triangle), whilst Capella will be in the Northeast. Given the UK latitude is between 50° and 60° (depending on your viewing location), you can work out that Deneb, Vega and Capella are all circumpolar (meaning they never set, so are always visible in the UK sky) since latitude + declination > 90°. Sirius will only be visible in the early morning sky in September (or in the evening 6 months later).	1

5.	B First we need to recognise that on the magnitude scale a large positive number is faint and a large negative number is bright. Hence, the brightest star is Alnilam (m = 1.69), and the second brightest is Alnitak (m = 1.77). Using the formula from the bottom of page 2 of the paper, $\frac{b_{\text{Alnilam}}}{b_{\text{Alnitak}}} = 10^{-0.4(m_{\text{Alnilam}} - m_{\text{Alnitak}})} = 10^{-0.4(1.69-1.77)} = 1.076$	1
6.	C Using the formula from the middle of page 2 of the paper $e = \sqrt{1 - \frac{b^2}{a^2}} \;\therefore\; b = a\sqrt{1-e^2} = 5.0\sqrt{1-0.80^2} = 3.0 \text{ au}$	1
7.	D For objects in the Solar System, if the semi-major axis is measured in au and the time is measured in years then Kepler's Third Law can be simplified as $T^2 = a^3$. If a = 5.0 au, then $T = \sqrt{(5.0^3)} = 11.2$ years.	1
8.	D The effect of atmospheric refraction is most pronounced very close to the horizon, meaning the positions of the Sun and Moon are offset by about 0.6°. This is bigger than their diameter, and so the upper limb of the Sun can still be seen even after its co-ordinates place it below the horizon, whilst the upper limb of the Moon can be seen even before its co-ordinates place it above the horizon.	1
9.	A Spectral lines simply add to give the overall spectrum of a mixture. Most stars are predominantly made of hydrogen and helium.	1
10.	B The maximum magnitude is about 22.2 and the minimum is about 24.5 (ignoring the large error bars), so using the formula from page 2 $\frac{b_{max}}{b_{min}} = 10^{-0.4(22.2-24.5)} = 8.3$ Since the amount of reflected light is proportional to the cross sectional area (which for an ellipse is πab), and assuming the comet is rotating around one of its minor axes (which is ≥ its shortest axis) then the ratio of the axes perpendicular to the rotation axis will be equal to the ratio of reflected light (so $b_{max} / b_{min} = a_{max} / a_{min}$).  Consequently, we are looking for an answer about 8 times bigger than 30 m, which is 240 m. Even taking into account the large error bars, option B is the only value close to the ones you would get.	1

Section B		10
11.	a)	**1**
	Apoapsis of Mercury: $$r_{\text{ap,Mer}} = a_{\text{Mer}}(1 + e_{\text{Mer}}) = 57.909(1 + 0.2056) = 69.8\ \text{Gm}$$	0.5
	Apoapsis of Neso: $$r_{\text{ap,Neso}} = a_{\text{Neso}}(1 + e_{\text{Neso}}) = 49.285(1 + 0.5714) = 77.4\ \text{Gm}$$	0.5
	So $r_{\text{ap,Neso}} > r_{\text{ap,Mer}}$	
	b)	**3**
	Kepler's Third Law (on page 2 of the paper) states that $\frac{a^3}{T^2} = \frac{GM}{4\pi^2}$ so $$\frac{M_{\text{Nep}}}{M_\odot} = \frac{\left(\frac{a_{\text{Neso}}}{a_{\text{Mer}}}\right)^3}{\left(\frac{T_{\text{Neso}}}{T_{\text{Mer}}}\right)^2}$$	1
	$$\therefore \frac{M_{\text{Nep}}}{M_\odot} = \frac{\left(\frac{49.285}{57.909}\right)^3}{\left(\frac{9740.73}{87.97}\right)^2}$$	1
	$$= 5.028 \times 10^{-5}$$	1
	[Note: so long as the units of the two values of *a* and *T* are the same, there is no need for unit conversions]	
	c)	**1**
	Using the result from part b): $$M_{\text{Nep}} = 5.028 \times 10^{-5} \times M_\odot = 5.028 \times 10^{-5} \times 1.99 \times 10^{30}$$ $$= 1.00 \times 10^{26}\ \text{kg}$$	1
	[This is very close to the accepted value of $M_{\text{Nep}} = 1.02 \times 10^{26}\ \text{kg}$]	

12.	a) Given that $$a = a_0(1+z)^{-1} \quad \text{and} \quad T = T_0(1+z) \quad \Rightarrow \quad \frac{a}{a_0} = \frac{T_0}{T}$$ and $$a \propto t^{2/3} \quad \Rightarrow \quad \frac{a}{a_0} = \left(\frac{t}{t_0}\right)^{2/3} = \frac{T_0}{T} \quad \therefore \quad t = t_0\left(\frac{T_0}{T}\right)^{3/2}$$ so $$t = 13.8 \times 10^9 \left(\frac{2.725}{3000}\right)^{3/2} = 3.78 \times 10^5 \text{ years}$$ [Alternatively allow the first mark for calculating the redshift of photon decoupling ($z = 1100$) **or** the scale factor at the time ($a = 9.08 \times 10^{-4}$), and the second for working out how much younger the Universe was ($t/t_0 = 2.74 \times 10^{-5}$)]	**3** 1 1 1
	b) $$t = t_0\left(\frac{T_0}{T}\right)^{3/2} \quad \therefore \quad T = T_0\left(\frac{t}{t_0}\right)^{-2/3}$$ $$= 2.725\left(\frac{13.8 - 4.5}{13.8}\right)^{-2/3} = 3.55 \text{ K}$$ (So it has dropped by less than 1 K in the whole history of the Earth!) [Allow 1 mark if they work out the temperature at an age of 4.5 Gyr, rather than 4.5 Gyr ago ($T = 5.75$ K)]	**2** 1 1

<table>
<tr><th>Section C</th><th></th><th>10</th></tr>
<tr><td>13.</td><td>a)

We need to combine Kepler's Third Law (on page 2 of the paper) with the Roche limit:
$$P_{\text{min}}^2 = \frac{4\pi^2}{GM_\star} a_{\text{min}}^3 = \frac{4\pi^2}{GM_\star}(2.44)^3 R_\star^3 \frac{\rho_\star}{\rho_p}$$
However the radius of the star is related to its mass and density via
$$\rho_\star = \frac{M_\star}{\frac{4}{3}\pi R_\star^3} \quad \therefore P_{\text{min}}^2 = \frac{4\pi^2}{GM_\star}(2.44)^3 R_\star^3 \frac{M_\star}{\frac{4}{3}\pi R_\star^3}\frac{1}{\rho_p}$$
$$\therefore P_{\text{min}}^2 = \frac{3\pi}{G}(2.44)^3 \frac{1}{\rho_p} \qquad \therefore P_{\text{min}} = \sqrt{\frac{3\pi(2.44)^3}{G\rho_p}}$$
[Note: the first half mark is for recognising to combine Kepler's 3rd Law with the given equation, the second half mark is for identifying how to eliminate the radius and density of the star. The final mark can be for either an expression for P_{min} or P_{min}^2, so long as they are simplified]</td><td>2

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<tr><td></td><td>b)

We need to rearrange the given equation to work out the mass of the planet for both types of composition. From this we can work out the density of the planet (since we know the radius is 0.61 R_E) and hence the minimum orbital period
$$\log\left(\frac{R}{R_E}\right) = 0.295\log\left(\frac{M}{M_E}\right) + \alpha \therefore M = 10^{\frac{\log(R/R_E)-\alpha}{0.295}} M_E$$
For the silicate (Si) case:
$$M = 10^{\frac{\log(R/R_E)-0.0286}{0.295}} M_E = 10^{\frac{\log(0.61)-0.0286}{0.295}} M_E = 0.150\, M_E$$
$$\rho_p = \frac{M}{\frac{4}{3}\pi R^3} = \frac{0.150 \times 5.97 \times 10^{24}}{\frac{4}{3}\pi(0.61 \times 6.37 \times 10^6)^3} = 3640 \text{ kg m}^{-3}$$
$$P_{\text{min}} = \sqrt{\frac{3\pi(2.44)^3}{G\rho_p}} = \sqrt{\frac{3\pi(2.44)^3}{6.67 \times 10^{-11} \times 3640}} = 23754 \text{ s} = 6.60 \text{ hrs}$$
Similarly, for the iron (Fe) case:
$$M = 0.438\, M_E$$
$$\rho_p = 10650 \text{ kg m}^{-3}$$
$$P_{\text{min}} = 3.86 \text{ hrs}$$
[Note: masses can be given in units of M_E or kg, but P_{min} <u>must</u> be given in hours. Masses and densities may not be explicitly calculated so accept valid alternative methods. Allow ecf where possible.]</td><td>6

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	c) It is an estimate (since to do the proper calculation is non-trivial) so students may approach this by either assuming that the percentage increases linearly with decreasing P_{min} from 0% for the silicate P_{min} to 100% for the iron Pmin, or assuming the percentage increases linearly with increasing ρ_p from 0% for the silicate ρ_p to 100% for the iron ρ_p. Accept other sensible approaches if clearly physically justified. Using P_{min}: $$P_{\text{min}} = 0.1768913 \text{ days} = 4.25 \text{ hours}$$ $$\therefore \frac{Fe}{M} = \frac{P_{\text{min}}(Si) - P_{\text{min}}}{P_{\text{min}}(Si) - P_{\text{min}}(Fe)} = \frac{6.60 - 4.25}{6.60 - 3.86} = 0.858 \therefore 85.8\%$$ So KOI 1843.03 **is** an iron planet **OR** Using ρ_p: $$P_{\text{min}} = 4.25 \text{ hrs} \therefore \rho_p = 8790 \text{ kg m}^{-3}$$ $$\therefore \frac{Fe}{M} = \frac{\rho_p - \rho_p(Si)}{\rho_p(Fe) - \rho_p(Si)} = \frac{8790 - 3640}{10650 - 3640} = 0.735 \therefore 73.5\%$$ So KOI 1843.03 **is not** an iron planet [Note: first mark is for the percentage, the second is for the conclusion. Do not give the second mark for a bald statement of is/is not an iron planet – there must be a calculation. For comparison purposes, Mercury is about 68% Fe and Earth is about 35%]	**2** 1 1 1 1

14.	a) We are told the period is half a day (= 12 hours) so we can use Kepler's Third Law to work out its orbital radius and hence its orbital speed, from which we can calculate the amount of time dilation $$a_{\text{GPS}} = \sqrt[3]{\frac{GM_E T^2}{4\pi^2}} = \sqrt[3]{\frac{6.67 \times 10^{-11} \times 5.97 \times 10^{24} \times (12 \times 3600)^2}{4\pi^2}} = 2.66 \times 10^7 \text{ m}$$ $$v = \frac{2\pi a_{\text{GPS}}}{T} = \frac{2\pi \times 2.66 \times 10^7}{12 \times 3600} = 3870 \text{ m s}^{-1}$$ $$\Delta t_{\text{SR}} = (1-\gamma)t_0 = \left(1 - \frac{1}{\sqrt{1 - \frac{3870^2}{(3.00 \times 10^8)^2}}}\right) \times (24 \times 3600) = -7.19 \times 10^{-6} \text{ s} \quad (\approx -7 \text{ µs})$$ [Note: a_{GPS} may not be explicitly calculated, so give that mark if the orbital speed is correct (required for the second mark). The value of Δt_{SR} must be to at least 2 s.f. to get the final mark. Most calculators can handle the bracketed expression directly, but using the binomial approximation gives the same value]	**3** 1 1 1
	b) Having already worked out a_{GPS} in the previous question, we can just plug the values directly into the equation given: $$\Delta t_{\text{overall}} = (\Gamma_{GPS} - \Gamma_E)t_0 = \left(\sqrt{1 - \frac{3GM_E}{a_{\text{GPS}}c^2}} - \sqrt{1 - \frac{2GM_E}{R_E c^2}}\right)t_0$$ $$= \left(\sqrt{1 - \frac{3 \times 6.67 \times 10^{-11} \times 5.97 \times 10^{24}}{2.66 \times 10^7 \times (3.00 \times 10^8)^2}} - \sqrt{1 - \frac{2 \times 6.67 \times 10^{-11} \times 5.97 \times 10^{24}}{6.37 \times 10^6 \times (3.00 \times 10^8)^2}}\right) \times (24 \times 3600)$$ $$= 3.84 \times 10^{-5} \text{ s} \quad \therefore \Delta t_{\text{overall}} = +38.4 \text{ µs}$$ A positive value for $\Delta t_{\text{overall}}$ means that time passes quicker on the satellite relative to people on the ground [Note: first mark is for careful substitution (allow ecf for value of a_{GPS}), second mark for a value that <u>must</u> be in µs, and third for a relevant statement. A calculation must be present for the third mark]	**3** 1 1 1

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The daily offset can simply be calculated by considering how far the radio beams will travel in the time difference between the satellite and the ground

$$\text{offset} = \Delta t_{\text{overall}} c = 38.4 \times 10^{-6} \times 3.00 \times 10^{8} = 11540\ \text{m}$$

[Note: The real situation is a little more complicated than a drift of 11 km a day. You need at least 4 satellites to get a lock at any location, and this drift would only be achieved if one satellite ignores relativity and the rest take it into account. If all of the satellites ignore relativity then they are in sync with each other and the offset largely cancels itself out. In practice, rather than messing with timing calculations, they broadcast their radio signals at (10.23 MHz – 4.55 mHz) in their rest frame so we receive them on Earth as exactly 10.23 MHz]</td><td>1

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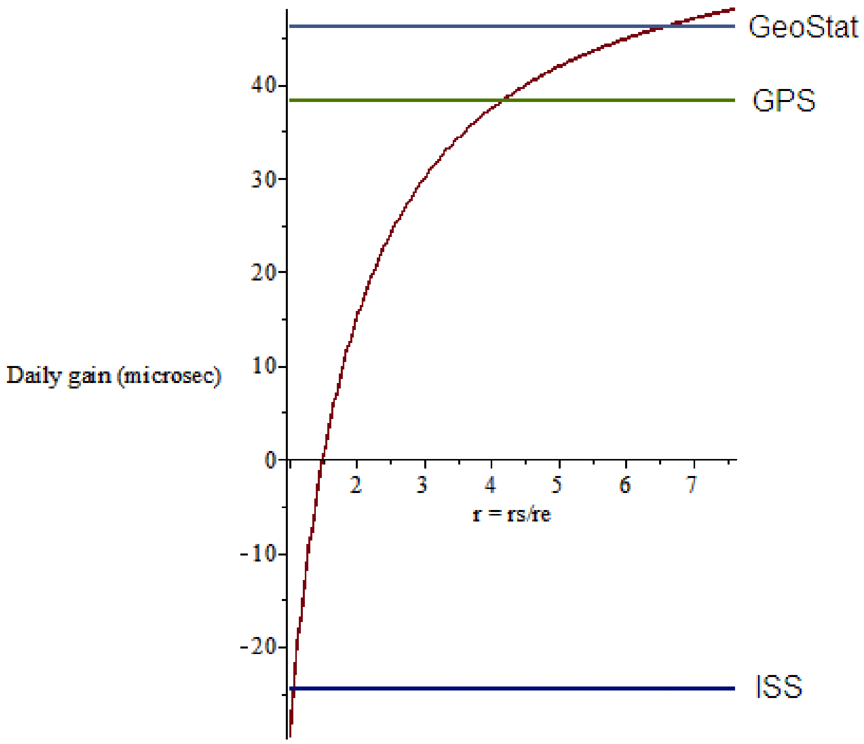
<tr><td></td><td>d)


Correct shape of curve (just the red line needed) with axes labelled

Crosses x-axis at $r = 1.5R_E$

Daily gain is approximately -30 μs at $r = R_E$ and 47 μs at $r = 7R_E$
(Accept graphs that start at $r = 0$, or are plotted with fractional change (i.e. $\Delta t_{\text{overall}}/t_0$) on the y-axis. Ignore any extension of the curve to orbital radii less than $r = R_E$. No need to have the horizontal lines in the above diagram)

[Note: this means special relativity effects are dominant in low Earth orbit, so astronauts on the ISS age slower than the rest of us!]</td><td>3

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