BAAO 2016/17 Solutions and Marking Guidelines

Note for markers:

- Answers to two or three significant figures are generally acceptable. The solution may give more in order to make the calculation clear.
- There are multiple ways to solve some of the questions; please accept all good solutions that arrive at the correct answer. If a candidate gets the final (numerical) answer then allow them all the marks for that **part** of the question (as indicated in red), so long as there are no unphysical / nonsensical steps or assumptions made.

Q1 - Martian GPS [Total = 20]

a. *Given that the Earth's sidereal day is 23h 56 mins, calculate the orbital radius of a GPS satellite. Express your answer in units of $R_\oplus$.*

$$T = \frac{1}{2} \times (23\text{h } 56\text{m}) = 718 \text{ mins} = 43080 \text{ s} \quad [1]$$

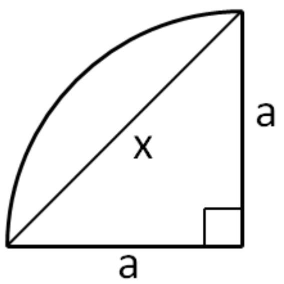
$$T^2 = \frac{4\pi^2}{GM} a^3 \quad \therefore a = \sqrt[3]{\frac{GM_\oplus T^2}{4\pi^2}} = \sqrt[3]{\frac{6.67\times10^{-11}\times5.97\times10^{24}\times43080^2}{4\pi^2}} \quad [1]$$

$$= 2.66 \times 10^7 \text{ m}$$

$$= 4.17 \text{ R}_\oplus \quad [1] \quad [3]$$

b. *How long would it take a radio signal to travel directly between a satellite and its closest neighbour in its orbital plane (assuming they're evenly spaced)? How far would a car on a motorway (with a speed of 30 m s^{-1}) travel in that time? [This can be taken to be a very crude estimate of the positional accuracy of the system for that car.].*

Closest neighbour in orbital plane should be 90° away (since evenly spaced) so can use Pythagoras



$$\therefore x = \sqrt{2a^2} = 3.76 \times 10^7 \text{ m} \quad [1]$$

Time for the signal to travel that distance:

$$t = \frac{x}{c} = \frac{3.76\times10^7}{3.00\times10^8} = 0.125 \text{ s} \quad [1] \quad [2]$$

Distance travelled by a car on a motorway in that time:

$$d = vt = 30 \times 0.125 = 3.76 \text{ m} \quad [1] \quad [1]$$

[In practice the positional accuracy of a GPS system is much harder to calculate; at low speeds it is typically a function of fluctuations and reflections of the signal within the atmosphere, as well as the presence of objects that might block the signal]

c. *Using suitable calculations, explore the viability of a 24-satellite GPS constellation similar to the one used on Earth, in a semi-synchronous Martian orbit, by considering:*

i. *Would the moons prevent such an orbit?*

$T = \frac{1}{2} \times (24\text{h } 37\text{m}) = 738.5 \text{ mins} = 44310 \text{ s}$ [1]

$T^2 = \frac{4\pi^2}{GM} a^3 \quad \therefore a = \sqrt[3]{\frac{GM_M T^2}{4\pi^2}} = \sqrt[3]{\frac{6.67\times10^{-11}\times6.42\times10^{23}\times44310^2}{4\pi^2}}$ [1]

$= 1.29 \times 10^7 \text{ m}$ [1] [3]

This is about ½ the distance to Diemos and about 3500 km (about R_M) away from Phobos so the moons should not provide a problem for any GPS satellite constellation [1] [1]

ii. *How would the GPS positional accuracy compare to Earth?*

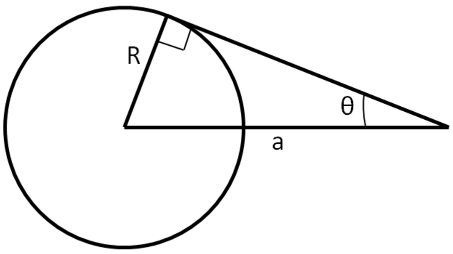
(Using similar reasoning to part b.)

$x = 1.82 \times 10^7 \text{ m}$ [1]

$t = 0.061 \text{ s}$ (or $d = 1.82 \text{ m}$) [1] [2]

The positional accuracy is about twice as good as on Earth [1] [1]

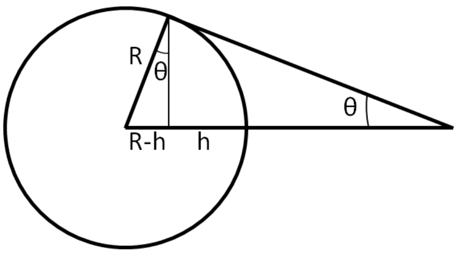
iii. *What would the receiving angle of each satellite's antenna need to be, and what would be the associated satellite footprint? By comparing these with the ones utilised by Earth's GPS, make a final comment on the viability of future Martian GPS.*



Receiving angle:

$2\theta = 2\sin^{-1}\left(\frac{R_M}{a}\right) = 2\sin^{-1}\left(\frac{3.39\times10^6}{1.29\times10^7}\right)$ [1]

$= 30.6°$ [1] [2]



The area of a 'zone' of a sphere is 2π*Rh* where *h* is the radial height of the zone. From the geometry of the situation:

$\sin\theta = \frac{R-h}{R} \quad \therefore h = R(1 - \sin\theta)$ [1]

$= 0.74 \text{ R}_M$ [1]

Fraction of surface area:

$\frac{\text{Area of zone}}{\text{Surface area of Mars}} = \frac{2\pi R_M (0.74 R_M)}{4\pi R_M^2} = 36.8\%$

[1] [3]

Receiving angle similar to Earth's so can use current GPS satellite technology [0.5]

Satellite footprint similar to Earth's so should get sufficient coverage [0.5]

Martian GPS system is viable [1] [2]

Q2 - Hohmann Transfer [Total = 20]

a. *Show that v_{orb} in low Earth orbit (LEO; about 200 km above the surface) is about 8 km s^{-1}. This is an estimate of the Δv the rockets need to provide for the spacecraft to reach LEO.*

$$v_{\text{orb}} = \sqrt{\frac{GM_\oplus}{R_\oplus + r}} = \sqrt{\frac{6.67\times10^{-11}\times5.97\times10^{24}}{6.37\times10^{6}+200\times10^{3}}} \quad [1]$$

$$= 7790 \text{ m s}^{-1} \quad (= 7.79 \text{ km s}^{-1} \approx 8 \text{ km s}^{-1}) \quad [1] \quad [2]$$

[In practice a Δv of 8 km s^{-1} assumes no external forces, but atmospheric drag can increase the necessary Δv by 1.3 – 1.8 km s^{-1}. When travelling between objects in space, however, such drag forces are absent and so the Δv calculated is much more accurate]

b. *Derive expressions for Δv_A and Δv_B by comparing their circular orbital speeds with their transfer orbit speeds. Simplify your final expressions to include G, $M_\odot$, r_A and r_B only.*

$$\Delta v_{\text{A}} = \sqrt{GM_\odot\left(\frac{2}{r_{\text{A}}} - \frac{1}{a}\right)} - \sqrt{GM_\odot\left(\frac{1}{r_{\text{A}}}\right)} = \sqrt{\frac{GM_\odot}{r_{\text{A}}}}\left(\sqrt{2 - \frac{r_{\text{A}}}{a}} - 1\right) \quad [1]$$

But $2a = r_A + r_B$

$$\therefore \Delta v_{\text{A}} = \sqrt{\frac{GM_\odot}{r_{\text{A}}}}\left(\sqrt{2 - \frac{2r_{\text{A}}}{r_{\text{A}}+r_{\text{B}}}} - 1\right) = \sqrt{\frac{GM_\odot}{r_{\text{A}}}}\left(\sqrt{\frac{2r_{\text{B}}}{r_{\text{A}}+r_{\text{B}}}} - 1\right) \quad [1] \quad [2]$$

Similarly:

$$\Delta v_{\text{B}} = \sqrt{\frac{GM_\odot}{r_{\text{B}}}}\left(1 - \sqrt{2 - \frac{2r_{\text{B}}}{r_{\text{A}}+r_{\text{B}}}}\right) = \sqrt{\frac{GM_\odot}{r_{\text{B}}}}\left(1 - \sqrt{\frac{2r_{\text{A}}}{r_{\text{A}}+r_{\text{B}}}}\right) \quad [1] \quad [1]$$

[These equations have been written so that the change in speed is positive, however give full credit for reversed signs (so long as they are consistent)]

c. *Approximating Mars' orbit as circular with a radius of 1.52 AU, calculate the Δv to go from Earth LEO to Mars i.e. $\Delta v = |\Delta v_A|+|\Delta v_B|$. Compare your answer to the Δv to reach Earth LEO.*

$$\Delta v_{\text{A}} = \sqrt{\frac{6.67\times10^{-11}\times1.99\times10^{30}}{1.49\times10^{11}}}\left(\sqrt{\frac{2\times1.52}{1+1.52}} - 1\right)$$

$$= 2935 \text{ m s}^{-1} \ (= 2.94 \text{ km s}^{-1}) \quad [1]$$

$$\Delta v_{\text{B}} = \sqrt{\frac{6.67\times10^{-11}\times1.99\times10^{30}}{1.52\times1.49\times10^{11}}}\left(1 - \sqrt{\frac{2\times1}{1+1.52}}\right)$$

$$= 2642 \text{ m s}^{-1} \ (= 2.64 \text{ km s}^{-1}) \quad [1]$$

$$\Delta v = 2.94 + 2.64$$

$$= 5.58 \text{ km s}^{-1} \quad [1] \quad [3]$$

This is less than the Δv to get into LEO [1] [1]

(So most of the effort needed in going to Mars simply comes from leaving Earth)

[The Δv calculated here would be for the spacecraft to enter a circular orbit around the Sun at the same distance as Mars, but this would not constitute landing – the extra Δv to get to the Martian surface increases the total for the whole transfer to roughly 8 km s^{-1} (same as to get into LEO). Other, more complicated routes can be taken, some of which offer substantial efficiencies so the Δv for the trip can be much lower (although the time taken to complete the manoeuvre will be longer, and the Δv to reach LEO will still be the biggest single step)]

d. *Derive an expression for the total time spent on the transfer orbit, t_H, and calculate it for an Earth to Mars transfer. Give your answer in months. (Use 1 month = 30 days).*

From Kepler's third law: $T^2 = \frac{4\pi^2}{GM}a^3$

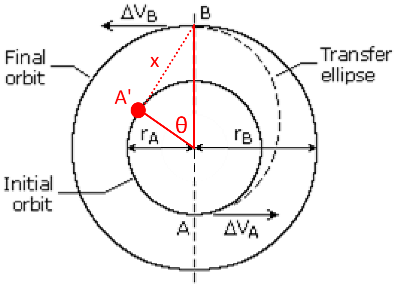
Since the spacecraft only covers half of the ellipse the time on the journey is half the period, and given that $2a = r_A + r_B$ then:

$$t_H = \pi\sqrt{\frac{(r_A+r_B)^3}{8GM_\odot}}$$ [1] [1]

$$= \pi\sqrt{\frac{((1+1.52)\times1.49\times10^{11})^3}{8\times6.67\times10^{-11}\times1.99\times10^{30}}} = 2.22\times10^7 \text{ s}$$ [1]

$$= 8.56 \text{ months}$$ [1] [2]

e. *Hence calculate the direct distance between Earth and Mars at the moment the spacecraft reaches Mars. How long would it take a radio message from the spacecraft to reach Earth?*



Initially Earth is at A. When the spacecraft reaches B (after 8.56 months), the Earth has moved round the Sun in its orbit and is now at A'.

Angle between A' and B:

$$\theta = \left(\frac{8.56-6}{12}\right)\times 360° = 76.7°$$ [1]

Using the cosine rule:

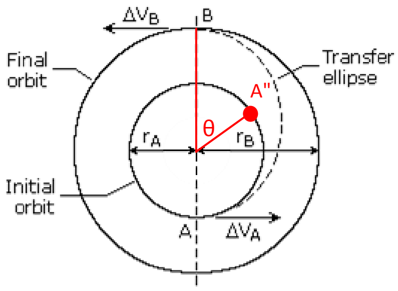
$$x^2 = r_A^2 + r_B^2 - 2r_A r_B\cos\theta$$

$$\therefore x = 2.41\times10^{11} \text{ m} \quad (= 1.62 \text{ AU})$$ [1] [2]

Since radio waves travel at the speed of light the time taken by the message is:

$$t = \frac{2.41\times10^{11}}{3.00\times10^8} = 803 \text{ s} \quad (= 13.4 \text{ minutes})$$ [1] [1]

[In practice, due to the eccentricity of Mars' orbit, the signal transmission time varies depending on the year the spacecraft was launched. When the rover Curiosity arrived at Mars the engineers described the landing as 7 minutes of terror, since the signal from the spacecraft would take 14 minutes to reach Earth but the time to transverse the Martian atmosphere was only 7 minutes (hence the process had to be completely automated)] *f. How long would any astronauts on board the spacecraft need to wait until they could use a Hohmann transfer orbit to return to Earth? Hence calculate the total duration of the mission.*



During the transfer the Earth moves by $180° + \theta = 257°$, so the spacecraft should launch from Mars when Earth is at position A" (exhibiting symmetry with when it arrived).

Since the planets move anti-clockwise in this diagram the angle covered by Earth from A' to A" is $360° - 2\theta = 206.5°$ [1]

From Kepler's third law, the period of Mars $= \sqrt{1.52^3} = 1.87$ years [1]

Therefore, the relative angular velocity of Earth if Mars' motion is subtracted out is:

$$\omega_{\text{rel}} = \frac{360°}{12} - \frac{360°}{1.87 \times 12} = 14.0° \text{ month}^{-1}$$ [1]

(allow any equivalent units e.g. 168° year^{-1}, 9.38×10^{-8} rad s^{-1} etc.)

Consequently, the time the astronauts need to wait for Earth to get from position A' to A" is:

$$\frac{206}{14.0} = 14.8 \text{ months} \quad (= 443 \text{ days})$$ [1] [4]

Thus the total duration of a return mission to Mars is:

$$t = 14.8 + (2 \times 8.56) = 31.9 \text{ months} \quad (= 956 \text{ days} = 2.66 \text{ years})$$ [1] [1]

[Shorter missions are possible, but would require a greater *Δv* and hence need much more fuel - any future mission will have to balance the cost (and mass) of more fuel on a fast trip with the cost (and mass) of more supplies on a slow trip]

Q3 - Starkiller Base [Total = 20]

a. *Assume the Sun was initially made of pure hydrogen, carries out nuclear fusion at a constant rate and will continue to do so until the hydrogen in its core is used up. If the mass of the core is 10% of the star, and 0.7% of the mass in each fusion reaction is converted into energy, show that the Sun's lifespan on the main sequence is approximately 10 billion years.*

Time on main sequence = total nuclear energy available / luminosity

$$t_{\mathrm{MS}} = \frac{E_{\mathrm{nuc}}}{\mathrm{L}_\odot} = \frac{\Delta m c^2}{\mathrm{L}_\odot} = \frac{f\epsilon \mathrm{M}_\odot c^2}{\mathrm{L}_\odot} = \frac{0.1\times0.007\times1.99\times10^{30}\times(3.00\times10^{8})^2}{3.85\times10^{26}} \quad [1]$$

$$= 3.26\times10^{17}\ \mathrm{s} = 1.03\times10^{10}\ \text{years} \quad [1] \quad [2]$$

$$(\approx 10\ \text{billion years})$$

b. *The Starkiller Base is able to stop nuclear fusion in the Sun's core*

i. *At its current luminosity, how long would it take the Sun to radiate away all of its gravitational binding energy? (This is an estimate of how long it would take to drain a whole star when radiatively charging the superweapon.)*

Time radiating energy = total gravitational binding energy / luminosity

$$t_{\mathrm{rad}} = \frac{U_\odot}{\mathrm{L}_\odot} = \frac{3G\mathrm{M}_\odot^2}{5\mathrm{R}_\odot \mathrm{L}_\odot} = \frac{3\times6.67\times10^{11}\times(1.99\times10^{30})^2}{5\times6.96\times10^{8}\times3.85\times10^{26}} \quad [1]$$

$$= 5.91\times10^{14}\ \mathrm{s}\ (= 18.7\ \mathrm{Myr}) \quad [1] \quad [2]$$

ii. *How does your value compare to the main sequence lifetime of the Sun calculated in part a.?*

This is much shorter than t_{MS} [1] [1]

iii. *Comment on whether there were (or will be) any events in the life of the Sun with a timescale of this order of magnitude.*

An event in the Sun's life that happened on a timescale of this order of magnitude is the gravitational collapse of the protostar before it joined the main sequence [1] [1]

[The Sun will also be on the asymptotic giant branch (AGB) for a similar order of magnitude of time - this is when the core is completely carbon / oxygen (but no longer undergoing fusion) and there is a spherical shell of helium burning happening just outside the core (with a shell outside that of hydrogen burning). Credit this answer too if a student mentions it.]

c. *In practice, the gravitational binding energy of the Earth is much lower than that of the Sun, and so the First Order would not need to drain the whole star to get enough energy to destroy the Earth. Assuming the weapon is able to channel towards it all the energy being radiated from the Sun's entire surface, how long would it take them to charge the superweapon sufficiently to do this?*

Time charging the weapon = total energy needed / rate of energy transfer

$$t_{\mathrm{charge}} = \frac{U_\oplus}{\mathrm{L}_\odot} = \frac{3G\mathrm{M}_\oplus^2}{5\mathrm{R}_\oplus \mathrm{L}_\odot} = \frac{3\times6.67\times10^{11}\times(5.97\times10^{24})^2}{5\times6.37\times10^{6}\times3.85\times10^{26}} \quad [1]$$

$$= 5.82\times10^{5}\ \mathrm{s}\ (= 6.7\ \text{days}) \quad [1] \quad [2]$$

(So it would only take a week to absorb enough energy from the Sun to destroy the Earth!)

d. *Taking the Starkiller Base's ice planet to have a diameter of 660 km, show that the Sun can be safely contained, even if it was fully drained.*

Need to work out the Schwarzschild radius for the Sun, and compare it to the size of the base

$$R_S = \frac{2GM_\odot}{c^2} = \frac{2\times6.67\times10^{-11}\times1.99\times10^{30}}{(3.00\times10^8)^2} = 2.95\times10^3\ \text{m}\ (= 2.95\ \text{km})$$ [1] [1]

(About 100 times) Smaller than the radius of the base ∴ the Sun can be safely contained [1] [1]

e. *The Starkiller Base wants to destroy all the planets in a stellar system on the far side of the galaxy and so drains 0.10 $M_\odot$ from the Sun to charge its weapon. Assuming that the U per unit volume of the Sun stays approximately constant during this process, calculate:*

i. *The new luminosity of the Sun.*

Need to use the mass-luminosity relation for main sequence stars ($L \propto M^4$) to work out the luminosity of a 0.9 $M_\odot$ star (since that is the new mass of the Sun)

$$L \propto M^4 \quad \therefore \quad \frac{L_{new}}{L_\odot} = \left(\frac{M_{new}}{M_\odot}\right)^4$$

$$\therefore \quad L_{new} = 0.9^4\ L_\odot = 0.66\ L_\odot\ (= 2.53\times10^{26}\ W)$$ [1] [1]

ii. *The new radius of the Sun.*

Energy density = U / V = constant

$$\frac{U}{V} = \frac{\frac{3GM^2}{5R}}{\frac{4}{3}\pi R^3} = \frac{9GM^2}{20\pi R^4} \propto \frac{M^2}{R^4} = \text{constant} \quad \therefore \quad \frac{M_\odot^2}{R_\odot^4} = \frac{M_{new}^2}{R_{new}^4} \quad \Rightarrow \quad \frac{M_\odot}{R_\odot^2} = \frac{M_{new}}{R_{new}^2}$$ [1]

$$R_{new} = \sqrt{\frac{M_{new}}{M_\odot}}\ R_\odot = \sqrt{0.9}\ R_\odot = 0.95\ R_\odot\ (= 6.60\times10^8\ \text{m})$$ [1] [2]

iii. *The new temperature of the surface of the Sun (current $T_\odot$ = 5780 K), and suggest (with a suitable calculation) what change will be seen in terms of its colour.*

Need to use the Stephan-Boltzmann Law to get the new temperature and then Wien's Law to determine the effect on the peak wavelength (and hence the colour)

$$L = 4\pi R^2\sigma T^4 \quad \therefore \quad \frac{L_{new}}{L_\odot} = \left(\frac{R_{new}}{R_\odot}\right)^2 \left(\frac{T_{new}}{T_\odot}\right)^4$$

$$\therefore \quad T_{new} = \sqrt[4]{\frac{\left(\frac{L_{new}}{L_\odot}\right)}{\left(\frac{R_{new}}{R_\odot}\right)^2}}\ T_\odot = \sqrt[4]{\frac{0.9^4}{0.9}}\ T_\odot$$ [1]

$$= 5340\ \text{K}\ (= 0.92\ T_\odot)$$ [1] [2]

$$\lambda_{peak,new} = \frac{2.90\times10^{-3}}{T_{new}} = \frac{2.90\times10^{-3}}{5340} = 5.43\times10^{-7}\ \text{m}\ (= 543\ \text{nm})$$ [1] [1]

This is a longer wavelength than the current peak (500 nm) so the Sun is **redder** [1] [1] *f. Assume that at the moment of destruction of the Starkiller Base the mass of the new star formed is equal to the mass drained from the Sun (0.10 $M_\odot$). Derive an expression for the main sequence lifetime in terms of stellar mass, and hence calculate the main sequence lifetime of this new star.*

We can combine the mass-luminosity relation with the expression we used in part a.

$$t_{\mathrm{MS}} \propto \frac{M}{L} \quad but \;\; L \propto M^4 \quad \therefore \quad t_{\mathrm{MS}} \propto \frac{M}{M^4} \propto M^{-3}$$ [1] [1]

$$\therefore t_{\mathrm{MS,new}} = \left(\frac{M_{\mathrm{new}}}{\mathrm{M}_\odot}\right)^{-3} t_{\mathrm{MS},\odot} = 0.1^{-3} \times 10^{10} \text{ years}$$ [1]

$$= 10^{13} \text{ years}$$ [1] [2]

Q4 - Hanny's Voorwerp [Total = 20]

a. *Given that Hubble's constant is measured as H_0 = 70 km s^{-1} Mpc^{-1}, calculate the distance to the galaxy (in Mpc).*

Need to turn the redshift into a recessional velocity and then combine with Hubble's Law

$$z = \frac{v}{c} \text{ (for small } z) \;\therefore\; v = zc = 0.05 \times 3.00 \times 10^5 = 15\,000 \text{ km s}^{-1} \quad [1]$$

$$d = \frac{v}{H_0} = \frac{15\,000}{70} = 214 \text{ Mpc} \quad [1] \quad [2]$$

b. *Calculate the power (luminosity) of the source required to completely ionize the Voorwerp (assumed to be spherical), given that the mass of a hydrogen atom is 1.67 × 10^{-27} kg and the ionization energy of hydrogen is 13.6 eV, where 1 eV = 1.60 × 10^{-19} J.*

Given we know the radius of the cloud (10 kpc) and the mass (10^{11} $M_\odot$) we can work out the number density of hydrogen atoms

$$S_* = Vn^2\alpha \quad \text{and} \quad n = \frac{M/m_H}{V} \quad \text{and} \quad V = \frac{4}{3}\pi R^3$$

$$\therefore S_* = \left(\frac{M}{m_H}\right)^2 \frac{\alpha}{V} \quad [1]$$

$$= \left(\frac{10^{11}\,M_\odot}{m_H}\right)^2 \frac{\alpha}{\frac{4}{3}\pi R^3} = \left(\frac{10^{11}\times 1.99\times 10^{30}}{1.67\times 10^{-27}}\right)^2 \frac{2.6\times 10^{-19}}{\frac{4}{3}\pi\times(10\times 10^3\times 3.09\times 10^{16})^3} \quad [1]$$

$$= 2.99 \times 10^{55} \text{ photons s}^{-1} \quad [1]$$

(watch that the units of α are converted correctly to SI)

The luminosity can then be calculated as we know the energy of each photon

$$L = S_* E_{\text{photon}} = 2.99 \times 10^{55} \times 13.6 \times 1.60 \times 10^{-19} \quad [1]$$

$$= 6.50 \times 10^{37} \text{ W} \quad [1] \quad [5]$$

(Allow full credit for interpreting the 10 kpc 'size' of the cloud to mean its diameter rather than its radius, giving S_* = 2.39 × 10^{56} photons s^{-1} and L = 5.20 × 10^{38} W)
[Working out S_* directly may prove difficult for some calculators as $(M/m_H)^2$ may exceed their largest power of ten, in which case students should work out $\sqrt{S_*}$ and then square it later.]

c. *The gravitational potential energy of the material falling to radius R, which in this case is a black hole with radius equal to the Schwarzschild radius, $R_S = 2GM/c^2$, at a mass accretion rate $\dot{m} \equiv \delta m/\delta t$, is converted into radiation with an efficiency of . Show that the power (luminosity) output of the SMBH is given by $L = \frac{1}{2}\eta\dot{m}c^2$.*

We know the gravitational potential energy of a particle of mass m at the Schwarzschild radius is the same as the kinetic energy it has gained moving from infinity to that point, so

$$E = \frac{GMm}{R_S} = \frac{GMm}{\frac{2GM}{c^2}} = \frac{1}{2}mc^2 \quad [1]$$

Given that a fraction η is converted into radiation and the given mass accretion rate then

$$L = \eta\frac{\delta E}{\delta t} = \eta \times \frac{1}{2}\frac{\delta m}{\delta t}c^2 = \frac{1}{2}\eta\dot{m}c^2 \quad [1] \quad [2]$$

(So the maximum energy you can get from a black hole is half the rest mass energy of the material falling in – this is a much more efficient process for generating energy than the 0.7% you get from nuclear fusion in stars, which in themselves are much more efficient than chemical reactions!)

d. *The typical mass accretion rate onto an active SMBH is ~2 $M_\odot$ yr^{-1} and the typical efficiency is η = 0.1. Calculate the typical luminosity of a quasar. Compare the luminosity of the quasar with the power needed to ionize the Voorwerp.*

Need to convert the mass accretion rate into kg s^{-1} and then put into the formula

$$L = \frac{1}{2}\eta\dot{m}c^2 \qquad = \frac{1}{2} \times 0.1 \times \left(\frac{2\times1.99\times10^{30}}{365.25\times24\times60\times60}\right) \times (3.00\times10^8)^2 \qquad [1]$$

$$= 5.68 \times 10^{38}\ \mathrm{W} \qquad [1] \qquad [2]$$

The luminosity of the quasar is high enough to ionize the Voorwerp [1] [1]

e. *Calculate the projected physical separation, r_p, between the galaxy and the Voorwerp.*

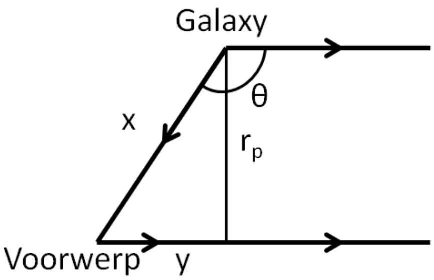
Since the angle is so small, we can use the small angle approximation for tan θ ≈ θ

$$20\ \text{arcseconds} = 9.70\times10^{-5}\ \text{radians} \qquad [0.5]$$

$$r_\mathrm{p} = \theta d = 9.70\times10^{-5}\times 214\ \mathrm{Mpc} \qquad [0.5]$$

$$= 20.8\ \mathrm{kpc}\ (= 6.42\times10^{20}\ \mathrm{m}) \qquad [1] \qquad [2]$$

f. *Derive an expression for the difference in the light travel time between photons travelling directly to Earth from the galaxy and photons reflected off the Voorwerp first. Give your formula as a function of r_p and θ, where θ is the angle between the lines of sight to the Earth and to the centre of the Voorwerp as measured by an observer at the centre of IC 2497. (For example θ = 90° would correspond to the galaxy and Voorwerp both being the exact same distance from the Earth, and so the projected distance r_p is therefore also the true distance between them.)*



Given the small angular separation we can treat the light rays from the galaxy to Earth and from the Voorwerp to Earth as essentially parallel, and so the difference in light travel time comes from the extra distance travelled in being reflected off the Voorwerp

Relevant diagram, suitably labelled [2]

Extra distance = x + y

$$x = \frac{r_p}{\cos(\theta-90^\circ)} = \frac{r_p}{\sin\theta} \qquad [1]$$

$$y = r_p\tan(\theta-90^\circ) = \frac{r_p}{-\tan\theta} \qquad [1]$$

$$\therefore\ \text{extra time} = \frac{\text{extra distance}}{c} = \frac{r_p}{c}\left(\frac{1}{\sin\theta} - \frac{1}{\tan\theta}\right) \qquad [1] \qquad [5]$$

$$= \frac{r_p}{c\sin\theta}(1-\cos\theta) \quad \text{(alternative form)}$$

[Allow any equivalent formula, for example expressing it in terms of csc θ and cot θ, so long as some attempt has been made to simplify it. It is quicker and simpler to derive if θ is assumed to be acute – we show it this way in case students see that the angle is obtuse from the next part of the question and want to have a consistent picture throughout]

g. *High precision measurements showed that the Voorwerp is slightly further away than the galaxy, and so θ = 125°. Use this with your expression from the previous part of the question to estimate an upper limit for the number of years that have passed since the quasar was last active.*

$$\Delta t = \frac{r_p}{c \sin\theta}(1 - \cos\theta) \qquad = \frac{20.8\times10^3\times3.09\times10^{16}}{3.00\times10^8\times\sin 125°}(1 - \cos 125°)$$

$$= 4.11 \times 10^{12}\ \text{s} \qquad [0.5]$$

$$\approx 130\ 000\ \text{years} \qquad [0.5] \quad [1]$$

(This is remarkably recent on astronomical timescales!)

Q5 - Imaging an Exoplanet [Total = 20]

a. *Calculate the maximum angular separation between the star and the planet, assuming a circular orbit. Give your answer in arcseconds (where 3600 arcseconds = 1°).*

$$a = \sqrt[3]{\frac{GM_*T^2}{4\pi^2}} = \sqrt[3]{\frac{6.67\times10^{-11}\times(0.123\times1.99\times10^{30})\times(11.186\times24\times60\times60)^2}{4\pi^2}} \quad [1]$$

$$= 7.28\times10^9 \text{ m} = (0.049 \text{ AU}) \quad [1]$$

$$\theta_{\max} = \frac{a}{d} = \frac{7.28\times10^9}{1.295\times3.09\times10^{16}} = 1.82\times10^{-7} \text{ radians} \quad [0.5]$$

$$= 0.038 \text{ arcseconds} \quad [0.5] \quad [3]$$

b. *Determine the luminosity of the star and hence calculate the flux received on the Earth (in W m^{-2}) from both the star and the planet. Use them to work out the contrast ratio and thus the apparent magnitude of the planet. Assume the planet reflects half of the incident light and that $\mathcal{M}_{\odot} = 4.83$.*

$$\mathcal{M} = m - 5\log\left(\frac{d}{10}\right) = 11.13 - 5\log\left(\frac{1.295}{10}\right) = 15.57 \quad [1]$$

$$L = 10^{\left(\frac{\mathcal{M}-\mathcal{M}_{\odot}}{-2.5}\right)} \text{L}_{\odot} = 10^{\left(\frac{15.57-4.83}{-2.5}\right)}\times3.85\times10^{26} = 1.95\times10^{22} \text{ W} \quad [1] \quad [2]$$

$$(= 5.07\times10^{-5} \text{ L}_{\odot})$$

$$f_{\text{star}} = \frac{L}{4\pi d^2} = \frac{1.95\times10^{22}}{4\pi\times(1.295\times3.09\times10^{16})^2} = 9.69\times10^{-13} \text{ W m}^{-2} \quad [1] \quad [1]$$

$$f_{\text{planet}} = \frac{f_{\text{incident}}\times\text{fraction reflected}\times\text{cross-sectional area of planet}}{4\pi d^2}$$

$$= \frac{\frac{L}{4\pi a^2}\times0.5\times\pi r_p^2}{4\pi d^2} = \frac{\frac{1.95\times10^{22}}{4\pi\times(7.28\times10^9)^2}\times0.5\times\pi\times(1.1\times6.37\times10^6)^2}{4\pi\times(1.295\times3.09\times10^{16})^2} \quad [1]$$

$$= 1.12\times10^{-19} \text{ W m}^{-2} \quad [1] \quad [2]$$

$$CR = \frac{f_{\text{star}}}{f_{\text{planet}}} = \frac{9.69\times10^{-13}}{1.12\times10^{-19}} = 8.64\times10^6 \quad [1] \quad [1]$$

$$\Delta m = 2.5\log(CR) = 2.5\log(8.64\times10^6) = 17.3 \quad [1]$$

$$\therefore m_{\text{planet}} = m_{\text{star}} + \Delta m = 11.13 + 17.3 = 28.5 \quad [1] \quad [2]$$

[Since we only know the minimum radius of the exoplanet it could be larger and hence brighter, however it may also reflect less than half the incident light from the star and so be fainter – in practice the numbers used here are an optimistic estimate and it is more likely to be fainter.]

(Accepted alternative methods: if they use Stephan-Boltzmann's Law then L = 5.88 × 10^{23} W, f_{star} = 2.92 × 10^{-11} W m^{-2}, and f_{planet} = 3.39 × 10^{-18} W m^{-2}, though the contrast ratio and magnitude should be the same. Also accept if they assume only the day side is able to reflect and hence the apparent magnitude brightens to 27.9) *c. Verify that the HST (which is diffraction limited since it's in space) would be sensitive enough to image the planet in the visible, but is unable to resolve it from its host star (take λ = 550 nm).*

$$\theta_{\mathrm{HST}} = \frac{1.22\lambda}{D} = \frac{1.22 \times 550 \times 10^{-9}}{2.4} = 2.80 \times 10^{-7} \text{ radians } (= 0.058 \text{ arcseconds}) \qquad [0.5]$$

Since $\theta_{HST} > \theta_{max}$ then the HST can't resolve it [0.5] [1]

Apparent magnitude of planet (28.5) is brighter (greater) than limiting magnitude (31) [1] [1]

d. Calculate the exposure time needed for a Keck II image of the exoplanet to have an SNR of 3. Assume that the telescope has perfect AO, is observed at the longest wavelength for which the planet can still be resolved from the star, all the received flux from the planet consists of photons of that longest wavelength , $\varepsilon = 0.1$ and $b = 10^9$ photons s^{-1} (so b >> f). Comment on your answer.

$$\lambda_{\mathrm{Keck}} = \frac{\theta_{\max} D}{1.22} = \frac{1.82 \times 10^{-7} \times 10}{1.22} = 1.49 \times 10^{-6} \text{ m} \qquad [1]$$

Photon flux:

$$f = \frac{f_{\mathrm{planet}}}{E_{\mathrm{photon}}} = \frac{f_{\mathrm{planet}}}{\frac{hc}{\lambda_{\mathrm{Keck}}}} = \frac{1.12 \times 10^{-19}}{\frac{6.63 \times 10^{-34} \times 3.00 \times 10^{8}}{1.49 \times 10^{-6}}} = 0.84 \text{ photons m}^{-2}\text{ s}^{-1} \qquad [1]$$

(SB-Law f_{planet} gives f = 25.4 photons m^{-2} s^{-1}, and day side reflection increases f by a factor of 2)

Since b >> f we can simplify the denominator of the SNR formula by ignoring the **first** term, so

$$t = \frac{SNR^2 \times b}{f^2 A^2 \epsilon} = \frac{3^2 \times 10^9}{0.84^2 \times (\pi \times 5^2)^2 \times 0.1} = 1.62 \times 10^9 \text{ s} \qquad [1] \quad [3]$$

This is really long (> 50 years!) so it is unlikely Keck will ever be able to directly image it [1] [1]

(alternative f gives 22.7 ks, and hence the conclusion is that it ***is*** feasible)

e. How long an exposure would JWST need in order to get the same SNR as Keck II, again if observed at the longest wavelength for which the planet can still be resolved from the star by the telescope? (Make similar assumptions about the received flux and use the same value of ε).

(using similar reasoning to the previous part of the question)

$$\lambda_{\mathrm{JWST}} = 9.70 \times 10^{-7} \text{ m} \qquad [1]$$

$$f = 0.55 \text{ photons m}^{-2}\text{ s}^{-1} \qquad [1]$$

(alternative method gives f = 16.5 photons m^{-2} s^{-1})

Since b << f we can simplify the denominator of the SNR formula by ignoring the **second** term, so

$$t = \frac{SNR^2}{f A \epsilon} = \frac{3^2}{0.55 \times (\pi \times 3.25^2) \times 0.1} = 4.96 \text{ s} \qquad [1] \quad [3]$$

(alternative method gives t = 0.16 s, and day side reflection decreases t by a factor of 2)

[This is much more reasonable – the optimistic assumptions we have made throughout this question mean this is just a lower limit and so the actual exposure time will be longer, perhaps several minutes to get a much higher SNR and to compensate for the fact for that the sensitivity is lower at 970 nm than we have quoted since it varies with wavelength. The main thing that would prevent direct observation of the exoplanet by JWST would be if the contrast ratio was much higher than calculated here as that would mean the star's light would need to be blocked out by the Near Infrared Camera's coronagraph – unfortunately this can only be used at wavelengths too long to resolve the system.]

END OF PAPER