BAAO 2019/20 Solutions and Marking Guidelines

Note for markers:

- Answers to two or three significant figures are generally acceptable. The solution may give more in order to make the calculation clear. Units should be present on final answers when appropriate.
- There are multiple ways to solve some of the questions; please accept all good solutions that arrive at the correct answer. If a candidate gets the final (numerical) answer then allow them all the marks for that part of the question (as indicated in red), so long as there are no unphysical / nonsensical steps or assumptions made.

Q1 – Apollo 11 [35 marks]

a. Ignoring the effects of air resistance, the weight of the rocket, and assuming 1-D motion only

i. Show that the thrust generated by the S-IC is about 3.3×10^7 N and hence calculate the acceleration experienced by the astronauts firstly at lift-off and secondly when the S-IC finishes its burn (ignore that the S-IC ignites a few seconds before lift off). Give your answer in units of g_0.

$$F = -I_{sp} g_0 \dot{m} = -263 \times 9.81 \times \frac{(135.6-2283.9)\times 10^3}{168} = 3.30 \times 10^7 \text{ N} \quad [1] \quad [1]$$

[Since this was a 'show that' question, students should have at least given to 3 s.f. Allow the mark whether they've done the change in the mass of S-IC (as shown) or in the change in the mass of the whole rocket (i.e. the numbers shown below for m_{tot})]

At lift off, $m_{\text{tot}} = 2283.9 + 483.7 + 121.0 + 49.7 = 2938.3$ tonnes

$$\therefore a = \frac{F}{m_{\text{tot}}} = \frac{3.30\times 10^7}{2938.3\times 10^3} = 11.23 \text{ m s}^{-2} = 1.14\, g_0 \quad [1] \quad [1]$$

At the end of S-IC, , $m_{\text{tot}} = 135.6 + 483.7 + 121.0 + 49.7 = 790.0$ tonnes

$$\therefore a = \frac{F}{m_{\text{tot}}} = \frac{3.30\times 10^7}{790.0\times 10^3} = 41.76 \text{ m s}^{-2} = 4.26\, g_0 \quad [1] \quad [1]$$

[Using only the mass of the S-IC (giving accelerations at lift off and end of S-IC of 1.47 g_0 and 24.8 g_0 respectively) loses both of these marks, but this conceptual error is only penalised here with full ecf allowed in the rest of the question]

ii. Determine the constant acceleration produced by the third stage (S-IVB) of the Saturn V rocket and hence calculate the total mass carried into the parking orbit at the end of the first burn.

Combining the given thrust equation with Newton's 2nd Law to determine m = f(t),

$$-I_{sp} g_0 \dot{m} = ma \quad \therefore \dot{m} = -\frac{a}{I_{sp} g_0} m \quad \therefore m = m_0 \exp\left(-\frac{a}{I_{sp} g_0} t\right) \quad [1]$$

Rearranging and solving for the initial and final masses during the whole S-IV burn,

$$a = \frac{\ln\left(\frac{m_0}{m}\right) I_{sp} g_0}{t} = \frac{\ln\left(\frac{(121.0+49.7)}{(13.2+49.7)}\right)\times 421 \times 9.81}{(147+347)} = 8.35 \text{ m s}^{-2}\ (= 0.85\, g_0) \quad [1] \quad [2]$$

Using the exponential equation to calculate the final mass at the end of the first burn,

$$m = (121.0 + 49.7)\exp\left(-\frac{8.35}{421\times 9.81} \times 147\right) = 126.8 \text{ tonnes} \quad [1] \quad [1]$$

[Full ecf available for ignoring the mass of the Apollo spacecraft (49.7 t), assuming the same conceptual error was penalised earlier, leading to a = 18.5 m s^{-2} and m = 62.6 t]

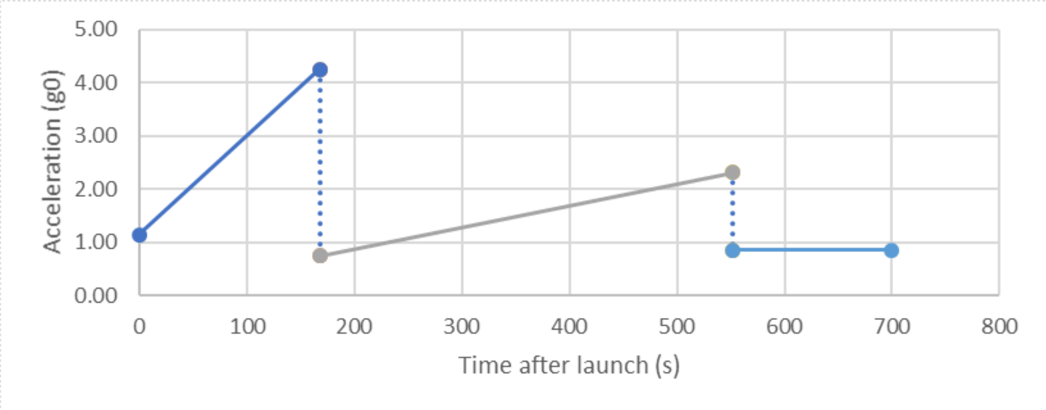
[This is close to the 118 tonnes that the Saturn V was designed to be able to carry into Low Earth Orbit, the largest of any rocket that has ever flown (although planned rockets launched in the coming decade will exceed this)]

iii. Sketch an acceleration-time graph of the journey from lift-off to reaching the parking orbit. Give accelerations in units of g_0. Assume the time between one stage finishing, detaching, and ignition of the next stage is negligible (i.e. you will have discontinuities in the graph at the end of each stage).

So far we have for S-IC $a_{\text{init}} = 1.14\ g_0$ and $a_{\text{final}} = 4.26\ g_0$, plus for S-IV $a = 0.85\ g_0$
Doing the same thing for S-II gives,

$$F = 4.77 \times 10^6\ \text{N} \quad \therefore a_{\text{init}} = 0.74\ g_0 \quad \text{and} \quad a_{\text{final}} = 2.31\ g_0$$ [2]

Plotting all of this information,



Graph has horizontal line for S-IV and only goes up to the end of first burn [1]
Graph has straight, diagonal lines for S-IC and S-II [1]
Appropriate axes labels (accel in g_0) [1] [5]
[The values of a (as summarised above) and t (0 s, 168 s, 552 s and 699 s) should either be listed in the working or clearly indicated in the sketch (since few students will try and do it to scale, unlike the graph above). Full ecf for incorrectly calculating the mass of the rocket gives for S-IC $a_{\text{init}} = 1.47\ g_0$ and $a_{\text{final}} = 24.8\ g_0$, for S-II $a_{\text{init}} = 1.01\ g_0$ and $a_{\text{final}} = 12.2\ g_0$, and for S-IV $a = 1.89\ g_0$. The lines are straight since we are ignoring the changing weight]

iv. By using your graph or otherwise, work out the speed of the rocket when reaching the parking orbit.

The area under the graph is the change in velocity of the rocket [1]
Working it out for each stage and adding them together,

$$\Delta v = 4451 + 5752 + 1227 = 11430\ \text{m s}^{-1}$$ [2] [3]

[Full ecf from the incorrect rocket masses leads to ~49250 m s^{-1}]

[It is worth noting that in under 12 minutes astronauts are in Earth orbit – staggeringly quick if you think about it – and they have to train to endure several seconds at an acceleration of about 4 g_0 (deeply uncomfortable!). If you integrate this graph twice you get the total change in displacement of the rocket at 4517 km – at this height above sea level and the calculated speed the orbit would not be bound (i.e. $E_{tot} > 0$), meaning that the effects of air resistance and weight of the rocket make a big difference, as does the fact the rocket rapidly rotates onto its side rather than following a radial path]

b. In reality, the effects of air resistance and the weight of the rocket are substantial. Once in the parking orbit it is travelling at 7.79 km s^{-1}.

i. What is its height above the Earth's surface (measured from sea level)? Give it in km.

For a circular orbit,

$$v = \sqrt{\frac{GM}{r}} \therefore r = \frac{GM}{v^2} = \frac{6.67\times10^{-11}\times5.97\times10^{24}}{(7.79\times10^3)^2} = 6.56\times10^6 \text{ m} \quad [1]$$

$$\therefore h = r - R_\oplus = 6.56\times10^3 - 6.37\times10^3 = 192 \text{ km} \quad [1] \quad [2]$$

[Must be in km for the final mark]

ii. The Apollo 11 spacecraft was in the parking orbit for 2 hours 32 mins 27 secs. How many revolutions of the Earth did it do?

From Kepler's third law, [Students may just use v and the orbital circumference to find T]

$$T^2 = \frac{4\pi^2}{GM}r^3 \quad \therefore T = \sqrt{\frac{4\pi^2}{6.67\times10^{-11}\times5.97\times10^{24}}\times(6.56\times10^6)^3} \quad [1]$$

$$= 5293 \text{ s} \quad [1]$$

$$\therefore n = \frac{(2\times3600)+(32\times60)+27}{5293} = \frac{9147}{5293} = 1.73 \text{ revolutions} \quad [1] \quad [3]$$

c. For the Hohmann transfer orbit (dashed line), find its semi-major axis and hence the duration of a translunar coast from A to B (expressed in hours and minutes).

The semi-major axis is half the total distance from A to B,

$$a_{Hoh} = \frac{1}{2}\left(h_A + R_\oplus + d_{E\to M} + R_M + h_B\right)$$

$$= \frac{1}{2}(334 + 6370 + 3.94\times10^5 + 1740 + 161)\times10^3 = 2.01\times10^8 \text{ m} \quad [1] \quad [1]$$

From Kepler's third law,

$$T^2 = \frac{4\pi^2}{GM}a_{Hoh}^3 \quad \therefore T = \sqrt{\frac{4\pi^2}{6.67\times10^{-11}\times5.97\times10^{24}}\times(2.01\times10^8)^3} \quad [1]$$

$$= 8.99\times10^5 \text{ s} \quad [1]$$

We will only travel half of the ellipse, so the time we want is half the period,

$$T_{Hoh} = \frac{1}{2}T = 4.50\times10^5 \text{ s} = 124 \text{ hr } 54 \text{ mins } (9 \text{ secs})$$ [must be in hr and min] [1] [3]

d. For the patched conics approach (solid lines):

i. Find the distance from the centre of the Earth to point C, and hence the semi-major axes of both ellipses.

At point C, the gravitational forces from both bodies are equal,

$$\frac{GM_\oplus}{x^2} = \frac{GM_M}{(d_{E\to M}-x)^2} \therefore \frac{d_{E\to M}-x}{x} = \sqrt{\frac{M_M}{M_\oplus}} \quad \therefore x = \left(1+\sqrt{\frac{M_M}{M_\oplus}}\right)^{-1} d_{E\to M} \quad [1]$$

$$\therefore x = \left(1+\sqrt{\frac{7.35\times10^{22}}{5.97\times10^{24}}}\right)^{-1}\times3.94\times10^8 = 3.55\times10^8 \text{ m} \quad [1] \quad [2]$$

Using a similar approach to the previous part of the question,

$$a_{AC} = \frac{1}{2}\left(h_A + R_\oplus + x\right) = 1.81\times10^8 \text{ m} \quad [1] \quad [1]$$

$$a_{BC} = \frac{1}{2}\left(h_B + R_M + x\right) = 2.06\times10^7 \text{ m} \quad [1] \quad [1]$$

[Allow the use of a first order binomial expansion for x giving $x = \left(1-\sqrt{\frac{M_M}{M_\oplus}}\right)d_{E\to M}$ leading to $x = 3.50\times10^8$ m, $a_{AC} = 1.78\times10^8$ m and $a_{BC} = 2.28\times10^7$ m.]

ii. Calculate the speed of the spacecraft at point A and point B. Give your answer in km s^{-1} and as a percentage of the escape speed of the spacecraft at that distance from the relevant closest gravitational body. Comment on what this implies for the eccentricity of the orbits.

Using the vis-viva equation given at the beginning of the paper,

$$v_A = \sqrt{GM_\oplus\left(\frac{2}{R_\oplus + h_A} - \frac{1}{a_{AC}}\right)}$$

$$= \sqrt{6.67\times10^{-11}\times5.97\times10^{24}\times\left(\frac{2}{(6370+334)\times10^3} - \frac{1}{1.81\times10^8}\right)}$$

$$= 10.8\ \text{km s}^{-1}$$ [1] [1]

As a percentage of the escape velocity,

$$v_{\text{esc},A} = \sqrt{\frac{2GM_\oplus}{R_\oplus + h_A}} = 10.9\ \text{km s}^{-1} \therefore v_A = 99.1\%\ v_{\text{esc},A}$$ [1] [1]

Similarly,

$$v_B = 2.22\ \text{km s}^{-1}$$ [1] [1]

$$v_{\text{esc},B} = 2.27\ \text{km s}^{-1} \therefore v_B = 97.7\%\ v_{\text{esc},B}$$ [1] [1]

Since these periapsis velocities are very close to the escape velocity, the eccentricities must be very close to 1 [1] [1]

[Do not award the marks for the percentage of the escape velocity if they have calculated the escape velocity at the surface of the body, rather than at the given altitude (i.e. R_x rather than $(R_x + h_x)$). The use of the binomial expansion in the previous part hardly affects these values (the only one changing at 3 s.f. is $v_B = 97.9\%\ v_{esc,B}$)]

iii. Determine the best estimate of the duration of the real Apollo 11 translunar coast. Give your answer in hours and minutes.

Using a similar approach to part c. where we only want half of the full ellipse,

$$T_{AC} = \frac{1}{2}\sqrt{\frac{4\pi^2}{6.67\times10^{-11}\times5.97\times10^{24}}\times(1.81\times10^8)^3} = 3.82\times10^5\ \text{s}$$ [1]

$$T_{BC} = \frac{1}{2}\sqrt{\frac{4\pi^2}{6.67\times10^{-11}\times7.35\times10^{22}}\times(2.06\times10^7)^3} = 1.33\times10^5\ \text{s}$$ [1]

The best estimate is exactly half of the sum of these two durations,

$$T_{\text{Apollo}} = \frac{1}{2}(T_{AC} + T_{BC}) = 2.58\times10^5\ \text{s} = 71\ \text{hr}\ 33\ \text{mins}\ (47\ \text{secs})$$ [1] [3]

[Must be in hr and min for the final mark. Use of the binomial expansion earlier gives 73 hr 36 mins 38 s]

[The real duration of the Apollo 11 translunar coast was 73 hr 5 mins 35 secs, so our second model of patched conics has done rather well compared to the Hohmann transfer. The main reasons for the differences are that the moon is not in a perfectly circular orbit around the Earth (so its gravitational influence varies during the coast more than our model allowed for) and we are assuming that the coast and the orbit are coplanar, which is not generally true. In later Apollo missions they used a manoeuvre during the coast to change the trajectory away from a simple translunar free-return trajectory, allowing for a greater range of landing sites – this means this simple model will only get close to the real durations for the early missions and for the later missions the full (non-trivial) calculations must be done.]

Q2 – Event Horizon Telescope and Super Massive Black Holes [35 marks]

a. Determine d_{max} for the observations of M87 (i.e. the solid lines in Fig 3) and hence θ_{min} if the EHT uses radio waves of frequency 230 GHz. Give your answer for d_{max} in km and θ_{min} in microarcseconds (1 degree = 3600 arcseconds).

Longest (solid) baseline from Fig. 3 is JCMT -> PV, so using 3-D Pythagoras,

$$d_{\max} = \sqrt{(x_{JCMT} - x_{PV})^2 + (y_{JCMT} - y_{PV})^2 + (z_{JCMT} - z_{PV})^2} \quad [1]$$

$$= 10907.93 \text{ km} \quad [1] \quad [2]$$

[Accept using SMA -> PV (giving d_{max} = 10907.86 km) for full marks, and d_{max} should be at least 3 s.f. and in km for the final mark. If the student uses PV -> SPT (the longest baseline, but a dashed line so not used for the M87 observation) giving d_{max} = 11388.83 km then they lose the final mark only]

Wavelength being used,

$$\lambda_{\text{obs}} = \frac{c}{f} = \frac{3.00\times10^8}{230\times10^9} = 1.30 \text{ mm} \quad [1]$$

$$\theta_{\min} = \frac{\lambda_{\text{obs}}}{d_{\max}} = \frac{1.30\times10^{-3}}{10907.93\times10^3} = 1.20 \times 10^{-10} \text{ rad} = 24.7 \text{ μas} \quad [1] \quad [2]$$

[Must be in microacrseconds (μas) for the final mark]

b. Assuming Sgr A* is a non-spinning black hole with mass 4.1×10^6 $M_\odot$ and at a distance of 8.34 kpc:

i. Derive the (unlensed) radius of the photon sphere, r_{ph}, in units of r_g, by considering a balance between the centripetal and (Newtonian) gravitational forces, but with the relativistic correction $v' = v\sqrt{1 - 2r_g/r}$ where v_0 is the classical velocity and r = r_{ph} when v = c.

Balancing centripetal forces and Newtonian gravity,

$$\frac{GMm}{r^2} = \frac{mv'^2}{r} \therefore v'^2 = \frac{GM}{r} = v^2\left(1 - \frac{2r_g}{r}\right) \quad [1]$$

Putting in the limit v = c and using the definition of r_g (= GM/c^2),

$$\frac{GM}{r_{ph}} = c^2\left(1 - \frac{2r_g}{r_{ph}}\right) \therefore \frac{GM}{c^2} = r_{ph}\left(1 - \frac{2r_g}{r_{ph}}\right) \quad \therefore r_g = r_{ph}\left(1 - \frac{2r_g}{r_{ph}}\right) \quad [1]$$

$$\therefore r_{ph} = 3r_g \quad [1] \quad [3]$$

ii. Determine the angular diameter (in microarcseconds) of the lensed photon sphere of Sgr A* and hence verify that the EHT can resolve it.

$$r_g = \frac{GM}{c^2} = \frac{6.67\times10^{-11}\times4.1\times10^6\times1.99\times10^{30}}{(3.00\times10^8)^2} = 6.05 \times 10^9 \text{ m} \quad [1]$$

Recognising that the lensed value we want is (3√3) r_g,

$$(3\sqrt{3})r_g = 3.14 \times 10^{10} \text{ m} \quad [1]$$

Determining the angular radius using trigonometry (and the small angle approximation),

$$\theta_r = \frac{(3\sqrt{3})r_g}{d} = \frac{3.14\times10^{10}}{8.34\times10^3\times3.09\times10^{16}} = 1.22 \times 10^{-10} \text{ rad} \quad (= 25.1 \text{ μas}) \quad [1]$$

Hence the angular diameter,

$$\theta_d = 2\theta_r = 2.44 \times 10^{-10} \text{ rad} = 50.3 \text{ μas} \quad [\theta_d > \theta_{\min} \therefore \text{resolvable}] \quad [1] \quad [4]$$

[Must be in μas and the angular diameter (not radius) for the final mark]

[This is larger than the measured angular diameter of M87* of 42 μas, meaning that Sgr A* is the easiest SMBH to *resolve*, but as seen later in the question it's not the easiest to *image*]

c. The angular diameter of M87* as determined from the images gained by the EHT (shown in Fig 4) is 42 microarcseconds, and the galaxy is 16.8 Mpc away from us. Determine the minimum and maximum possible masses of the SMBH in units of $M_\odot$.

The angular radius is,

$$\theta_r = 21\ \mu\text{as} = 1.02 \times 10^{-10}\ \text{rad}$$

The lensed physical radius is,

$$r_g' = d\theta_r = 16.8 \times 10^6 \times 3.09 \times 10^{16} \times 1.02 \times 10^{-10} = 5.29 \times 10^{13}\ \text{m} \qquad [1]$$

Looking at the minimum mass limit (i.e. the non-spinning case),

$$r_g' = (3\sqrt{3})r_{g,\min} \therefore r_{g,\min} = \frac{5.29\times10^{13}}{3\sqrt{3}} = 1.02 \times 10^{13}\ \text{m} \qquad [1]$$

$$\therefore M_{\min} = \frac{r_{g,min}c^2}{G} = 1.37 \times 10^{40}\ \text{kg} = 6.90 \times 10^9\ M_\odot \qquad [1] \quad [3]$$

Using the other lensed r_g limit gives for the maximum mass,

$$M_{\max} = 1.48 \times 10^{40}\ \text{kg} = 7.42 \times 10^9\ M_\odot \qquad [1] \quad [1]$$

[The first three marks are for any correct mass (i.e. done in either order), with the fourth for the other one. Final masses must be given in $M_\odot$]

d. Determine r_{ISCO} for a non-spinning black hole. Give your answer in units of r_g.

Want $\frac{dE}{dr} = 0$ given $E = mc^2(1-2r_gr^{-1})(1-3r_gr^{-1})^{-1/2}$

Applying the product rule, $\frac{d}{dr}(uv) = \frac{du}{dr}v + \frac{dv}{dr}u$,

$$u = 1 - 2r_gr^{-1} \qquad v = (1-3r_gr^{-1})^{-1/2}$$

$$\frac{du}{dr} = 2r_gr^{-2} \qquad \frac{dv}{dr} = -\frac{3}{2}r_gr^{-2}(1-3r_gr^{-1})^{-3/2} \qquad [1]$$

$$\therefore \frac{dE}{dr} = mc^2\left[2r_gr^{-2}(1-3r_gr^{-1})^{-1/2} - \frac{3}{2}r_gr^{-2}(1-3r_gr^{-1})^{-3/2}(1-2r_gr^{-1})\right] \qquad [1]$$

Making the expression in the square bracket equal zero (since m > 0),

$$2r_gr^{-2}(1-3r_gr^{-1})^{-1/2} = \frac{3}{2}r_gr^{-2}(1-3r_gr^{-1})^{-3/2}(1-2r_gr^{-1}) \qquad [1]$$

$$2(1-3r_gr^{-1}) = \frac{3}{2}(1-2r_gr^{-1})$$

$$\therefore r = 6r_g \qquad \text{(so } r_{ISCO} = 6r_g\text{)} \qquad [1] \quad [4]$$

[First mark is for differentiating the expressions for u and v correctly, second mark for having the correct form of dE/dr, third mark for setting the relevant part of the expression to zero, fourth mark for the final answer. Give full credit to use of the quotient rule or other valid approaches]

e. Taking the mass of M87* as $6.5 \times 10^9\ M_\odot$:

i. Determine the period of a particle in the ISCO of M87* for the a = 1, a = -1 and a = 0 (i.e. non-spinning) cases. Give your answer in days.

$$r_g = \frac{GM}{c^2} = \frac{6.67\times10^{-11}\times6.5\times10^9\times1.99\times10^{30}}{(3.00\times10^8)^2} = 9.59 \times 10^{12}\ \text{m}$$

For the a = 1 case,

$$\omega^2 = \frac{GM}{\left(2r_g^{3/2}\right)^2} = \frac{6.67\times10^{-11}\times6.5\times10^9\times1.99\times10^{30}}{\left(2(9.59\times10^{12})^{3/2}\right)^2} = 2.45 \times 10^{-10}\ \text{rad}^2\ \text{s}^{-2} \qquad [1]$$

$$T_{a=1} = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{2.45\times10^{-10}}} = 4.02 \times 10^5\ s = 4.65\ \text{days} \qquad [1] \quad [2]$$

Similarly,

$T_{a=0} = 34.15$ days [Allow ecf on this one only] [1] [1]

$T_{a=-1} = 60.42$ days [All periods must be in days] [1] [1]

[First one correctly calculated is worth two marks, the rest are one mark (done in any order)]

ii. One of the bright patches in Fig 4 seemed to move a quarter of the way around the ring between April 5 and 10 (from the left hand side to the bottom). Could it be attributable to gas moving in the ISCO? If so, is the spin likely to be positive or negative?

Quarter of a period in 5 days, so orbital period = 20 days

This is in the valid ISCO range (so could be gas moving in the ISCO) [1] [1]

This corresponds to a **positive** spin [1] [1]

[Allow full ecf on conclusions consistent with their earlier values]

iii. Determine the minimum and maximum ISCO periods for Sgr A* and hence suggest a possible reason why M87 has been imaged first, even though Sgr A* has a larger angular diameter, given that each 'exposure' with the EHT was 7 mins long (with multiple exposures from each observing run added together for the final image from each night).

Minimum ISCO period when a = 1, so using a similar approach to e.(i),

$\omega^2 = 6.15 \times 10^{-4}\ \mathrm{rad^2\ s^{-2}}$ $\quad \therefore T_{\min} = 253\ \mathrm{s}\ (= 4.22\ \mathrm{mins})$ [1] [1]

Maximum ISCO period when a = -1, $\quad \therefore T_{\max} = 3293\ \mathrm{s}\ (= 54.88\ \mathrm{mins})$ [1] [1]

Possibility that bright objects / gas clouds appear smeared out during exposures [1] [1]

[Allow difficult to determine time evolution of the SMBH on the timescale of the ISCO / any other relevant point from identifying that the period of the ISCO might be similar to the duration of the exposure (unlike with M87*)]

f. A particle close to M87* moves directly from r_{ISCO} to r_{ph} (and subsequently into the black hole). What is the extra distance travelled by it due to the curvature of spacetime, as described in Fig 5? Give your answer in au, and assume M87* is non-spinning.

In flat space,

$\Delta l_{\text{flat}} = r_{\text{ISCO}} - r_{ph} = 6r_g - 3r_g = 3r_g$ [Allow ecf] [1]

In curved space,

$$\Delta l = \int_{3r_g}^{6r_g} \left(1 - \frac{2r_g}{r}\right)^{-1/2} \mathrm{d}r$$

Using the following consecutive substitutions,

$u = \frac{r}{2r_g} \therefore \mathrm{d}u = \frac{1}{2r_g}\mathrm{d}r$ $\quad \therefore \Delta l = 2r_g \int_{3/2}^{3} \left(1 - \frac{1}{u}\right)^{-1/2} \mathrm{d}u$ [1]

$t^2 = 1 - \frac{1}{u} \therefore 2t\,\mathrm{d}t = \frac{1}{u^2}\,\mathrm{d}u = (1 - t^2)^2\,\mathrm{d}u$ $\quad \therefore \Delta l = 4r_g \int_{1/\sqrt{3}}^{\sqrt{2}/\sqrt{3}} \frac{1}{(1-t^2)^2}\,\mathrm{d}t$ [1]

Using difference of squares followed by partial fractions,

$$\frac{1}{(1-t^2)^2} = \frac{1}{(1-t)^2(1+t)^2} = \frac{2-t}{4(1-t)^2} + \frac{2+t}{4(1+t)^2} = \frac{1}{4}\left[\frac{1}{1-t} + \frac{1}{(1-t)^2} + \frac{1}{1+t} + \frac{1}{(1+t)^2}\right] \quad [1]$$

$$\therefore \Delta l = r_g \int_{1/\sqrt{3}}^{\sqrt{2}/\sqrt{3}} \frac{1}{1-t} + \frac{1}{(1-t)^2} + \frac{1}{1+t} + \frac{1}{(1+t)^2}\,\mathrm{d}t$$

$$\therefore \Delta l = r_g \left[\ln\left(\frac{1+t}{1-t}\right) + \frac{1}{1-t} - \frac{1}{1+t}\right]_{1/\sqrt{3}}^{\sqrt{2}/\sqrt{3}} \quad [1]$$

$$= 4.1424 r_g \quad [1]$$

So extra distance $= \Delta l - \Delta l_{\text{flat}} = 1.1424 r_g$ [so this is a big effect!]

$= 1.1424 \times 9.59 \times 10^{12} = 1.10 \times 10^{13}\ \mathrm{m} = 73\ \mathrm{au}$ [1] [7]

[Accept any alternative method of integrating the function, **including using numerical methods**. The final answer must be in au for the final mark. To help with ecf for incorrect limits, the integral as a function of the original variable is $\Delta l = \left[r\left(1 - \frac{2r_g}{r}\right)^{1/2} + r_g \ln\left(r\left(1 - \frac{2r_g}{r}\right)^{1/2} - r_g - r\right)\right]_{r_2}^{r_1}$ so the irrational number should be $\left(2\sqrt{6} - \sqrt{3} - \ln(2 + \sqrt{3}) + \ln(5 + 2\sqrt{6})\right) = 4.1424$ ...]

Q3 – Inspirals of Exoplanets Near Tidal Destruction [30 marks]

a. Show that WTS-2b has an orbital radius of ~1.4 a_{Roche}.

Using the given formula for a_{Roche},

$$a_{\text{Roche}} = 2.16 R_P \left(\frac{M_\star}{M_P}\right)^{1/3} = 2.16 \times 1.36 \times 7.15 \times 10^7 \times \left(\frac{0.820\times1.99\times10^{30}}{1.12\times1.90\times10^{27}}\right)^{1/3}$$

$$= 1.92 \times 10^9 \text{ m} \quad [1]$$

Using Kepler's third law,

$$P^2 = \frac{4\pi^2}{GM} a^3 \quad \therefore \; a = \sqrt[3]{\frac{GM}{4\pi^2} P^2}$$

$$= \sqrt[3]{\frac{6.67\times10^{-11}\times0.820\times1.99\times10^{30}}{4\pi^2} \times (1.0187 \times 24 \times 3600)^2} \quad [1]$$

$$= 2.77 \times 10^9 \text{ m} \quad [1]$$

$\therefore a = 1.44\, a_{\text{Roche}}$ [since 'show that' must be at least 3 s.f.] [1] [4]

b. Given the apparent magnitude of WTS-2 in the visible is m = 16.14 and the absolute magnitude of the Sun in the same part of the EM spectrum is $\mathcal{M}_\odot$ = 4.83:

i. Calculate the luminosity of the star, $L_\star$. Give your answer in units of $L_\odot$.

Since the absolute magnitude is the apparent magnitude at a fixed distance of 10 pc,

$$\frac{b_1}{b_0} = 10^{-0.4(m_1 - m_0)} \text{ and } b_{10\text{ pc}} = \frac{L}{4\pi(10\text{ pc})^2} \Rightarrow \frac{L_1}{L_0} = 10^{-0.4(\mathcal{M}_1 - \mathcal{M}_0)} \quad [1]$$

$$\mathcal{M}_\star = m_\star - 5\log\left(\frac{d}{10}\right) = 16.14 - 5\log\left(\frac{1.03\times10^3}{10}\right) = 6.08 \quad [1]$$

$$\frac{L_\star}{L_\odot} = 10^{-0.4(\mathcal{M}_\star - \mathcal{M}_\odot)} = 10^{-0.4(6.08-4.83)} \quad \therefore L_\star = 0.317\, L_\odot \quad [1] \quad [3]$$

[Final answer must be in $L_\odot$ for the final mark]

ii. Hence work out the radius of the star, $R_\star$. Give your answer in units of $R_\odot$.

Using Wien's displacement law,

$$T = \frac{B}{\lambda_{\max}} = \frac{2.90\times10^{-3}}{580\times10^{-9}} = 5000 \text{ K} \quad [1]$$

Using the Stephan-Boltzmann Law,

$$L_\star = 4\pi R_\star^2 \sigma T^4 \;\therefore R_\star = \sqrt{\frac{L_\star}{4\pi\sigma T^4}} = \sqrt{\frac{0.317\times3.85\times10^{26}}{4\pi\times5.67\times10^{-8}\times5000^4}} \quad [1]$$

$$= 5.24 \times 10^8 \text{ m} \quad = 0.753\, R_\odot \quad [1] \quad [3]$$

[Final answer must be in $R_\odot$ for the final mark]

iii. Show that the incident flux (in W m^{-2}) on the planet is ~10^3 times larger than what we receive on Earth from the Sun.

$$b_{\text{Earth}} = \frac{L_\odot}{4\pi(1\text{ au})^2} = \frac{3.85\times10^{26}}{4\pi\times(1.50\times10^{11})^2} = 1.36\times10^3 \text{ W m}^{-2} \quad [1]$$

$$b_P = \frac{L_\star}{4\pi a^2} = \frac{0.317\times3.85\times10^{26}}{4\pi\times(2.77\times10^9)^2} = 1.26\times10^6 \text{ W m}^{-2} \quad [1]$$

$\therefore b_P = 930\, b_{\text{Earth}}$ [since 'show that' must be at least 2 s.f.] [1] [3]

[The consequence of such a large incident flux is that the surface of this planet is incredibly hot – given it is also tidally locked (so the same hemisphere always faces the star) it is modelled to have a maximum day-side surface temperature as high as ~2000 K]

c. Derive an equation for τ, showing $\tau \propto a^8$. Hint: n is a function of a due to Kepler's third law.

$$n^2 = \left(\frac{2\pi}{P}\right)^2 = \frac{GM_\star}{a^3} \quad [1]$$

$$\therefore \frac{\dot{a}}{a} = 6k_2\Delta t\frac{M_P}{M_\star}\left(\frac{R_\star}{a}\right)^5 n^2 = 6k_2\Delta t\frac{M_P}{M_\star}\left(\frac{R_\star}{a}\right)^5\frac{GM_\star}{a^3}$$

$$\therefore \dot{a} = 6k_2\Delta tGM_PR_\star^5\frac{1}{a^7} \quad [1]$$

Separating variables and integrating,

$$\int_0^a a^7\,\mathrm{d}a = \int_0^\tau 6k_2\Delta tGM_PR_\star^5\,\mathrm{d}t \quad [1]$$

$$\frac{1}{8}a^8 = 6k_2\Delta tGM_PR_\star^5\tau \qquad \therefore \tau = \frac{a^8}{48k_2\Delta tGM_PR_\star^5} \quad [1] \quad [4]$$

[We know that a is actually decreasing as time increases, so removing the modulus sign from the original given equation should give a negative sign, however the limits of the integral have been swapped around so that the value for τ evaluates to a positive number – do not penalise any student getting a negative expression for τ]

[This incredibly steep dependence on a is why we can treat the time to get from the initial value of a to zero as essentially the same as the time taken to get to a_{Roche} (when the planet is destroyed)]

d. For the planet WTS-2b, assuming $Q'_\star = 10^6$:

i. Show that $\tau \propto \frac{Q'_\star}{n}\frac{M_\star}{M_P}\left(\frac{a}{R_\star}\right)^5$, finding the constant of proportionality (as a fraction), and hence the remaining lifetime of the planet. Give your answer in Myr. [1 Myr = 10^6 years]

$$Q'_\star = \frac{3}{4k_2\Delta tn} \therefore k_2\Delta t = \frac{3}{4Q'_\star n}$$

$$\tau = \frac{a^8}{48k_2\Delta tGM_PR_\star^5} = \frac{a^8}{48\left(\frac{3}{4Q'_\star n}\right)GM_PR_\star^5} = \frac{a^8Q'_\star n}{36GM_PR_\star^5} = \frac{Q'_\star na^3}{36M_PG}\left(\frac{a}{R_\star}\right)^5 \quad [1]$$

$$= \frac{Q'_\star n\left(\frac{GM_\star}{n^2}\right)}{36M_PG}\left(\frac{a}{R_\star}\right)^5 \quad [1]$$

$$= \frac{1}{36}\frac{Q'_\star}{n}\frac{M_\star}{M_P}\left(\frac{a}{R_\star}\right)^5$$ [so const = 1/36] [1] [3]

[First mark is for correctly substituting in the expression for $Q'_\star$, the second for use of Kepler's third law to cancel G and introduce $M_\star$, and the third for the value of the constant. Simply stating the constant without showing the rest of the algebraic expression cancelling into the required form loses at least the second mark.]

So we can now calculate the remaining lifetime,

$$\tau = \frac{1}{36}\frac{Q'_\star}{n}\frac{M_\star}{M_P}\left(\frac{a}{R_\star}\right)^5 = \frac{1}{36}\frac{10^6}{(2\pi/(1.0187\times24\times3600))}\frac{0.820\times1.99\times10^{30}}{1.12\times1.90\times10^{27}}\left(\frac{2.77\times10^9}{5.24\times10^8}\right)^5 \quad [1]$$

$$= 1.24\times10^{15}\ \text{s} = 39.4\ \text{Myr}$$ [must be in Myr] [1] [2]

[Allow full ecf for the student's values of a and $R_\star$ and be generous with errors that may have arose from rounding]

[This is unsurprisingly rather short on astronomical timescales – this planet really is close to destruction!]

ii. Show that $T_{shift} \propto \frac{T^2}{\tau}$, finding the constant of proportionality, and hence verify that T_{shift} is measurable (i.e. > 5 s) if T = 10 years. Hint: Using the chain rule, dn/dt = dn/da ȧ

$$\frac{\dot{a}}{a} = 6k_2\Delta t \frac{M_P}{M_\star}\left(\frac{R_\star}{a}\right)^5 n^2 = 6\frac{3}{4Q'_\star n}\frac{M_P}{M_\star}\left(\frac{R_\star}{a}\right)^5 n^2$$

$$\therefore \dot{a} = \frac{9}{2}\frac{n}{Q'_\star}\frac{M_P}{M_\star}\frac{R_\star^5}{a^4} \qquad [1]$$

$$n = \sqrt{\frac{GM_\star}{a^3}} = \sqrt{GM_\star}a^{-3/2} \qquad [1]$$

$$\therefore \frac{\mathrm{d}n}{\mathrm{d}a} = -\frac{3}{2}\sqrt{GM_\star}a^{-5/2} = -\frac{3}{2}\frac{n}{a} \qquad [2]$$

$$\frac{\mathrm{d}n}{\mathrm{d}t} = \left(\frac{\mathrm{d}n}{\mathrm{d}t}\right)\dot{a} = -\frac{3}{2}\frac{n}{a} \times \frac{9}{2}\frac{n}{Q'_\star}\frac{M_P}{M_\star}\frac{R_\star^5}{a^4} = \frac{27}{4}n^2\frac{M_P}{M_\star}\left(\frac{R_\star}{a}\right)^5\frac{1}{Q'_\star} \qquad [1]$$

Putting this into the given formula,

$$T_{\mathrm{shift}} = \frac{1}{2n}T^2\left(\frac{dn}{dt}\right) = \frac{1}{2n}T^2 \times \frac{27}{4}n^2\frac{M_P}{M_\star}\left(\frac{R_\star}{a}\right)^5\frac{1}{Q'_\star} = \frac{27}{8}T^2\frac{n}{Q'_\star}\frac{M_P}{M_\star}\left(\frac{R_\star}{a}\right)^5 \qquad [1]$$

But we found $\tau = \frac{1}{36}\frac{Q'_\star}{n}\frac{M_\star}{M_P}\left(\frac{a}{R_\star}\right)^5$ so,

$$T_{\mathrm{shift}} = \frac{3}{32}\frac{T^2}{\tau}$$ [so constant = 3/32] [1] [7]

Evaluating the expression for T = 10 years,

$$T_{shift} = \frac{3}{32}\frac{(10\times365\times24\times3600)^2}{1.24\times10^{15}} = 7.50 \text{ s}$$ [so > 5 s therefore measurable] [1] [1]

[Allow full ecf from their value of τ. Allow use of 365.25 days in a year]

[WTS-2b is one of a number of exoplanets that could produce measurable changes in their period over the next decade (see the right panel in Fig 6 in the paper), and hence are an incredibly useful way of constraining the value of $Q'_\star$, which in itself will greatly inform stellar models]