BAAO 2021/22 Solutions and Marking Guidelines

Note for markers:

- Answers to two or three significant figures are generally acceptable. The solution may give more in order to make the calculation clear. Units should be present on final answers when appropriate.
- There are multiple ways to solve some of the questions; please accept all good solutions that arrive at the correct answer. Students getting the answer in a $\boxed{\text{box}}$ will get all the marks available for that calculation / part of the question (as indicated in red), so long as there are no unphysical / nonsensical steps or assumptions made (students may not explicitly calculate the intermediate stages and should not be penalised for this so long as their argument is clear).

Q1 – Sunrise in Oxford [30 marks]

a. Consider an observer in Oxford (ϕ = +51.8°) on the June solstice.

i. Calculate the bearing of sunrise. Take the Sun to be a point source and ignore any atmospheric effects.

Sunrise corresponds to $h = 0$ [1]

Putting this in the given formula:

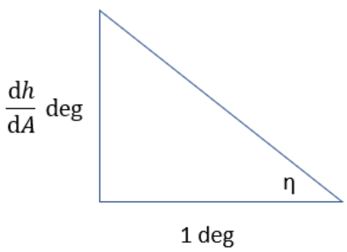
$$0 = -(90° - \phi)\cos A + \delta$$

$$\therefore \cos A = \frac{\delta}{90° - \phi} = \frac{23.44°}{90° - 51.8°} = 0.614 \quad \therefore \boxed{A = 52.1°}$$ [1] [2]

ii. By calculating the angle the solar path makes with the horizon, η, at sunrise for both the solstice and the equinox, estimate the duration of sunrise on the solstice if sunrise takes 3 mins 26 secs on the equinox. Assume the same solar angular velocity in both cases.

Using the given hint to differentiate the formula,

$$\frac{\mathrm{d}h}{\mathrm{d}A} = (90° - \phi)\sin A$$ [1]



Considering the sunrise angle, η, it is clear that

$$\tan\eta = \frac{\mathrm{d}h}{\mathrm{d}A}$$

Evaluating dh/dA at A = 90° (for the equinox) and at A = 52.1° (for the June solstice) gives

$$\eta_{eq} = \tan^{-1}(90° - 51.8) = \boxed{33.7°}$$ [1]

$$\eta_{sol} = \tan^{-1}[(90° - 51.8°)\sin 52.1°] = \boxed{27.8°}$$ [1] [3]

The Sun needs to travel vertically through some (fixed) angle Δh to set, so considering the vertical component of its angular velocity, ω

$$t = \frac{\Delta h}{\omega \sin\eta} \qquad \therefore \frac{t_{sol}}{t_{eq}} = \frac{\Delta h/(\omega \sin \eta_{\mathrm{sol}})}{\Delta h/(\omega \sin \eta_{\mathrm{eq}})} = \frac{\sin\eta_{eq}}{\sin\eta_{sol}} \therefore t_{sol} = \frac{\sin\eta_{eq}}{\sin\eta_{sol}} t_{eq}$$ [1]

Putting in our values for the setting angles,

$$t_{sol} = \frac{\sin 33.7°}{\sin 27.8°} t_{eq} = 1.19 t_{eq} = \boxed{4 \text{ min } 5 \text{ s}}$$ [1] [2]

[If the student wrongly assumes that the solstice sunset is shorter than at equinox and hence have the fraction of sin η the wrong way round (leading to 2 min 53 s) lose 4th mark but ecf on 5th; same penalty if using the ratio of dh/dA instead of the ratio of sin(tan^{-1}(dh/dA)) giving 4 min 21 s. If both mistakes are made (resulting in 2 min 43 s) subtract an additional mark. Allow 1 ecf mark for $\eta_{eq} = 38.2°$ if some justification given]

b. Considering just the bearing of sunrise, suggest (with qualitative justification only) which of the following situations the simple model will be the best approximation for the precise model:
A) a pole at solstice; B) a pole at equinox; C) the equator at solstice; or D) the equator at equinox.

When $h = 0°$, the simple model becomes $\delta = (90° - \phi)\cos A$
Similarly, the precise model becomes $\sin\delta = \cos\phi\cos A$
$\therefore$ need $\sin\delta \approx \delta$ so need δ small enough to use small angle approximation $\therefore$ equinox
and need $\cos\phi = \sin(90° - \phi) \approx (90° - \phi)$
$\therefore$ need $(90° - \phi)$ to be small $\therefore \phi \approx 90° \therefore$ pole
The best option is therefore $\boxed{\text{B (a pole at equinox)}}$ [3] [3]
[Answer without justification gets no marks. Lose 1 mark for each flaw in justification]

c. Reconsider the Oxford observer at the June solstice, but this time use the two equations of the precise model. Ignore any atmospheric effects.

i. Calculate the bearing of sunrise and the duration of the day (in hours and minutes), taking the Sun to be a point source.

When $h = 0°$, the precise model becomes $\sin\delta = \cos\phi\cos A$ [1]
[This mark may be awarded from working in part b. if not explicitly present here]

$$\therefore \cos A = \frac{\sin\delta}{\cos\phi} = \frac{\sin 23.44°}{\cos 51.8°} = 0.643 \qquad \therefore \boxed{A = 50.0°}$$ [1] [2]

Using the second given formula from the precise model with $h = 0°$

$$0 = \sin\phi\sin\delta + \cos\phi\cos\delta\cos H$$

$$\therefore \cos H = -\tan\phi\tan\delta$$ [1]

$$= -\tan 51.8°\tan 23.44° = -0.551 \qquad \therefore H = -123.4°$$ [1]

The day length is the time it takes the Sun to go from $-123.4°$ to $+123.4°$

$$\therefore \text{day length} = \frac{\Delta H}{360°} \times 24^{\text{h}}$$ [1]

$$= \frac{2\times 123.4°}{360°} \times 24^{\text{h}} = 16.46\text{ hr} = \boxed{16\text{ hr }27\text{ min}}$$ [1] [4]

[Due to the symmetry of sunrise and sunset, do not penalise giving only a positive value of H. Must be expressed in hours and minutes for the final mark]

[This shows the simple model has done rather well for determining the bearing of sunrise with a less than 5% error. It is clear that sunrise happens in the North East, not the East, on the June solstice (and, by symmetry, in the South East on the December solstice)]

ii. Calculate the duration of sunrise (in minutes and seconds), assuming a solar angular diameter of 0.525°.

The value of h now corresponds to the angle the centre of the solar disc makes with the horizon, so sunrise starts (h_-) when h is 1 solar angular radius below the horizon, and finishes (h_+) when it is 1 solar angular radius above the horizon

$$\therefore h_- = -\frac{0.525°}{2} = -0.2625° \quad \text{and} \quad h_+ = +\frac{0.525°}{2} = +0.2625°$$ [1]

Using the second given formula we can find out the corresponding solar hour angle,

$$\cos H_- = \frac{\sin h_- - \sin\phi\sin\delta}{\cos\phi\cos\delta}$$

$$= \frac{\sin(-0.2625°) - \sin 51.8°\sin 23.44°}{\cos 51.8°\cos 23.44°} = -0.559 \qquad \therefore H_- = -124.0°$$ [1]

Similarly, $H_+ = -122.9°$ [Again, no penalty for positive H values] [1]

$$\therefore t = \frac{\Delta H}{360°} \times 24^{\text{h}} = \frac{1.11°}{360°} \times 24^{\text{h}} = 0.074\text{ hr} = 4.44\text{ min} = \boxed{4\text{ min }26\text{ s}}$$ [1] [4]

[This is within 1 second of the real value (ignoring atmospheric effects), with the discrepancy coming only from not using a more precise value for the solar angular diameter. The simple model has not performed as well here as it did for the bearing, being about 8% off compared to 5% for the bearing – this extra discrepancy is mostly down to the fact that ω is slightly different at the equinox and solstice due to the elliptical shape of the Earth's orbit]

d. This exam is being taken on 24th January and is 3 hours long.

i. Estimate the solar declination on this date.

Given we are told that the declination varies sinusoidally from -23.44° on 21st Dec to +23.44° on 21st June, a reasonable approximation for δ (in degrees) would be

$$\delta = -23.44\cos\left(\frac{\text{days since December solstice}}{1\text{ year}} \times 360°\right) \qquad [1]$$

Since there are 31 days in December, 24th January is 34 days since the solstice [1]

$$\therefore \delta_{today} = -23.44\cos\left(\frac{34}{365} \times 360°\right) = \boxed{-19.54°} \qquad [1] \qquad [3]$$

[Accept other reasonable models and use of 365.25 days for 1 year. The 2nd mark is for the number of days since the winter solstice – allow attempts based upon the number of days since / before the June solstice. No marks for use of a linear model]

[The real solar declination is -19.17°, so this is a good estimate]

ii. Hence, calculate the latitude where today's day length is equal to the exam length, taking the Sun to be a point source.

If the day length is 3 hours, then when $h = 0$ need $H = -1.5^{\text{h}} = -22.5°$ [1]

Using a similar approach to c. (i) [No marks for using $H = -45°$]

$$\cos H = -\tan\phi\tan\delta$$

$$\therefore \tan\phi = -\frac{\cos H}{\tan\delta} = -\frac{\cos(-22.5°)}{\tan(-19.54°)} = 2.60 \qquad \therefore \boxed{\phi = 69.0°} \qquad [1] \qquad [2]$$

iii. What is the latitude with the longest sunrise today? Give its duration in minutes and seconds.

The longest possible sunrise will occur when it only just finishes at solar noon

$\therefore h = h_+$ when $H_+ = 0°$ [Allow diagram or qualitative statement] [1]

Putting this into the second given equation and using a suitable trigonometric identity

$$\sin h_+ = \sin\phi\sin\delta + \cos\phi\cos\delta$$
$$= \cos(\phi - \delta) \qquad [1]$$
$$\therefore \phi = \cos^{-1}(\sin h_+) + \delta$$
$$= \cos^{-1}(\sin 0.2625°) - 19.54° = \boxed{70.2°} \qquad [1] \qquad [3]$$

We can use a similar approach to c. (ii) to work out H_- (we already know $H_+ = 0°$) and hence duration of the sunrise:

$$\cos H_- = \frac{\sin h_- - \sin\phi\sin\delta}{\cos\phi\cos\delta} \qquad \text{[Again, no penalty for positive H values]}$$

$$= \frac{\sin(-0.2625°) - \sin 70.2°\sin(-19.54°)}{\cos 70.2°\cos(-19.54°)} = 0.971 \qquad \therefore H_- = -13.8° \qquad [1]$$

$$\therefore t = \frac{\Delta H}{360°} \times 24^{\text{h}} = \frac{13.8°}{360°} \times 24^{\text{h}} = 0.917\text{ hr} = 55.04\text{ min} = \boxed{55\text{ min }3\text{ s}} \qquad [1] \qquad [2]$$

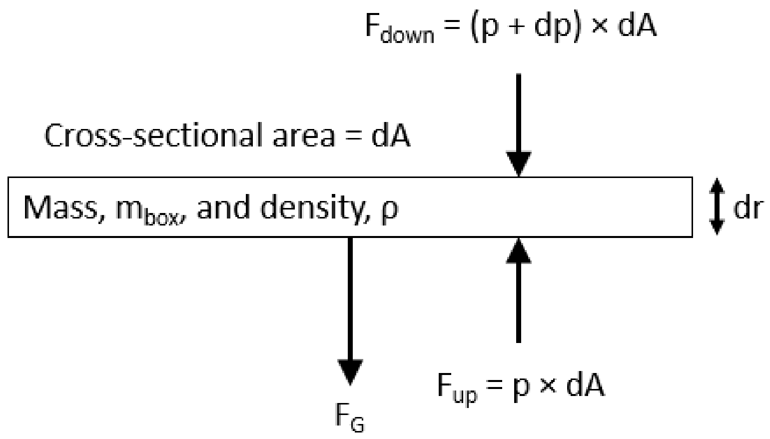
[This means the sunrise lasts half a day, and is almost one third of the length of this exam!]

Q2 – Stellar Structure [35 marks]

a. Let r denote distance from the centre of a star. We define the variables ρ(r), p(r) and T(r) to be the density, pressure and temperature at radius r respectively, and m(r) to be the mass enclosed within radius r. We will now try and derive an estimate for the pressure at the centre of the Sun.

i. By considering forces on a box of height dr at radius r, show that $dp/dr = -\rho Gm/r^2$.

Suitable diagram (e.g. side on, as below) showing all the relevant forces [1]



By balancing forces

$$F_{up} = F_G + F_{down}$$

$$\therefore p\mathrm{d}A = \frac{Gmm_{box}}{r^2} + (p + \mathrm{d}p)\mathrm{d}A$$

$$= \frac{Gm(\rho \mathrm{d}A\mathrm{d}r)}{r^2} + (p + \mathrm{d}p)\mathrm{d}A \qquad [1] \quad [2]$$

The dA and p on both sides cancel, leaving

$$\mathrm{d}p = -\frac{Gm\rho\mathrm{d}r}{r^2} \quad \therefore \frac{\mathrm{d}p}{\mathrm{d}r} = -\rho\frac{Gm}{r^2} \quad \text{(as required)}$$

[Since it was a 'show that' question, there is no credit for the final formula, just the working]

ii. We can get a good estimate of the central pressure if we use m as our independent variable rather than r. Derive an expression for dm/dr in terms of r and ρ, and hence express dp/dm in terms of m and r.

Considering a single spherical shell of density ρ and thickness dr, its mass is given as

$$\mathrm{d}m = \rho \times \text{volume} = \rho \times 4\pi r^2 \mathrm{d}r \qquad \therefore \boxed{\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi\rho r^2} \qquad [1] \quad [1]$$

Applying the chain rule with this result

$$\frac{\mathrm{d}p}{\mathrm{d}m} = \frac{\mathrm{d}p}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}m} = -\rho\frac{Gm}{r^2} \times \frac{1}{4\pi\rho r^2} \qquad \therefore \boxed{\frac{\mathrm{d}p}{\mathrm{d}m} = -\frac{Gm}{4\pi r^4}} \qquad [1] \quad [1]$$

iii. Assuming that the pressure at the surface, p_s, is negligible compared to the pressure at the centre of the Sun, p_c, the edge of the core is at r = 0.20 $R_\odot$ and encloses a mass of m = 0.35 $M_\odot$, and that dp/dm is constant throughout the star and equal to the value at the edge of the core, calculate a value for p_c.

Evaluating dp/dm at the edge of the core

$$\frac{\mathrm{d}p}{\mathrm{d}m} = -\frac{G(0.35\,M_\odot)}{4\pi(0.20\,R_\odot)^4}$$

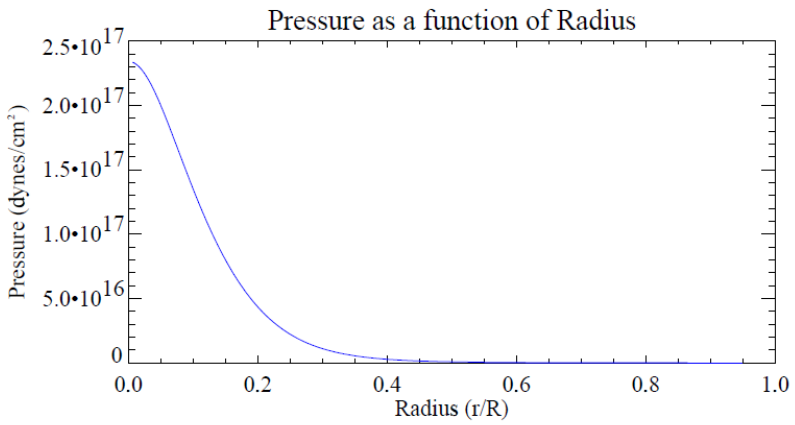
$$= -\frac{6.67\times10^{-11}\times(0.35\times1.99\times10^{30})}{4\pi(0.20\times6.96\times10^{8})^4} = -9.85\times10^{-15}\ \mathrm{Pa\ kg^{-1}} \qquad [1]$$

Since dp/dm is a constant

$$\frac{\mathrm{d}p}{\mathrm{d}m} = \frac{\Delta p}{\Delta m} = \frac{p_s - p_c}{M_\odot} \approx -\frac{p_c}{M_\odot} \qquad [1]$$

$$\therefore p_c = -\frac{\mathrm{d}p}{\mathrm{d}m}M_\odot = 9.85\times10^{-15}\times1.99\times10^{30} = \boxed{1.96\times10^{16}\ \mathrm{Pa}} \qquad [1] \qquad [3]$$

[This is rather close to the real value of 2.34 × 10^{16} Pa, which is very large compared to values we are used to e.g. it is about the same as 10^{11} atmospheres! The graph below shows how it varies as a function of radius (where 1 dyne/cm^2 = 10 Pa) justifying $p_s \approx 0$]



b. The Sun is composed predominantly of ionized hydrogen and helium, with approximate mass fractions X = 0.70 and Y = 0.30 respectively (taken to be constant throughout the Sun).

i. Show that the kinetic energy per unit mass of the solar plasma is the formula given.

Expressing the mass fractions in terms of the number of each atom, N, and the mass of each atom (ignoring electrons)

$$X = \frac{m_H N_H}{M_\odot} = \frac{m_p N_H}{M_\odot} \quad \text{and} \quad Y = \frac{m_{He} N_{He}}{M_\odot} = \frac{4 m_p N_{He}}{M_\odot} \qquad [1]$$

In a fully ionised plasma, each hydrogen atom contributes two particles (1 nucleus and 1 electron) whilst each helium atom contributes three particles (1 nucleus and 2 electrons) so the total number of particles is

$$N = 2N_H + 3N_{He} \qquad [1]$$

In thermodynamic equilibrium, all the particles will have the same average KE, so to get the KE per unit mass we need to divide the average KE by the average mass of a particle

$$u = \frac{\frac{3}{2}k_B T}{M_\odot / N} \qquad [1]$$

$$= \frac{3k_B T}{2M_\odot}(2N_H + 3N_{He}) = \frac{3k_B T}{2M_\odot}\left(2\frac{XM_\odot}{m_p} + 3\frac{YM_\odot}{4m_p}\right) \qquad [1] \qquad [4]$$

The $M_\odot$ cancel to leave $u = \frac{3k_B T}{2m_p}\left(2X + \frac{3}{4}Y\right)$ (as required)

[Since it was a 'show that' question, there is no credit for the final formula, just the working]

ii. Using the Virial Theorem, and given $E_G \approx GM_\odot^2/R_\odot$, estimate the Sun's mean temperature.

Combining the integral form for the LHS and the given expression for E_G with the non-integral form of the RHS, our expression for the Virial Theorem becomes

$$\int_0^{M_\odot} u\,dm = -\frac{1}{2}E_G$$

$$\therefore \int_0^{M_\odot} \frac{3k_BT}{2m_p}\left(2X + \frac{3}{4}Y\right) dm = \frac{GM_\odot^2}{2R_\odot}$$ [ignore any minus sign on RHS] [1]

Comparing to the form we are aiming for

$$\langle T_\odot \rangle = \frac{1}{M_\odot}\int_0^{M_\odot} T\,dm = \frac{\frac{GM_\odot}{2R_\odot}}{\frac{3k_B}{2m_p}\left(2X+\frac{3}{4}Y\right)} = \frac{GM_\odot m_p}{3R_\odot k_B\left(2X+\frac{3}{4}Y\right)}$$ [1]

$$= \frac{6.67\times10^{-11}\times1.99\times10^{30}\times1.67\times10^{-27}}{3\times6.96\times10^{8}\times1.38\times10^{-23}\left(2\times0.70+\frac{3}{4}\times0.30\right)} = \boxed{4.73\times10^6\ \mathrm{K}}$$ [1] [3]

c. Considering the evaluated equations for τ, R, and q we can use this with the measured luminosity of the Sun to get a new estimate for the central temperature.

i. Considering the simplified equation for q and assuming that the core has a mass of 0.35 $M_\odot$, throughout which T and ρ are constant, and that the Sun's luminosity is equal to the power produced by the p-p chain fusion processes occurring within its core, estimate the central temperature. [You are given that $u = 3p_c/2\rho_c$.]

Combining the given expression for $u = 3p_c/2\rho_c$ with the one from b. (i)

$$u = \frac{3k_BT}{2m_p}\left(2X + \frac{3}{4}Y\right) = \frac{3p_c}{2\rho_c} \qquad \therefore \rho_c = \frac{p_c m_p}{k_BT\left(2X+\frac{3}{4}Y\right)}$$ [1]

Assuming all of the power output of the Sun comes from the reaction in the core

$$L_\odot = q \times 0.35M_\odot$$ [1]

Equating the given expression for q with this gives

$$0.251\rho_c X^2 T_6^{-2/3} e^{-33.80T_6^{-1/3}} = \frac{L_\odot}{0.35M_\odot}$$

$$\therefore 0.251\frac{p_c m_p}{k_BT\left(2X+\frac{3}{4}Y\right)} X^2 T_6^{-2/3} e^{-33.80T_6^{-1/3}} = \frac{L_\odot}{0.35M_\odot}$$ [1]

$$\therefore T_6^{-5/3} e^{-33.80T_6^{-1/3}} = \frac{10^6 L_\odot k_B\left(2X+\frac{3}{4}Y\right)}{0.251X^2 p_c m_p (0.35M_\odot)}$$ [1]

$$= \frac{10^6\times3.85\times10^{26}\times1.38\times10^{-23}\left(2\times0.70+\frac{3}{4}\times0.30\right)}{0.251\times0.70^2\times1.96\times10^{16}\times1.67\times10^{-27}\times0.35\times1.99\times10^{30}}$$

$$= 3.08\times10^{-9}$$ [1]

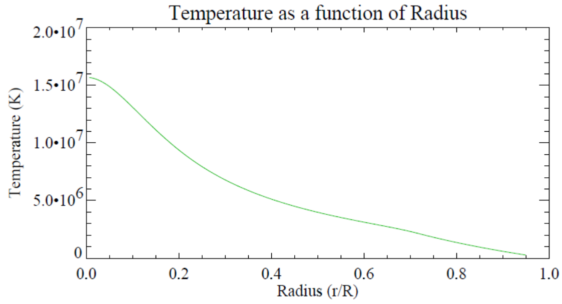
Using an appropriate iterative method with $T_6 = 4.73$ as the starting point [1]

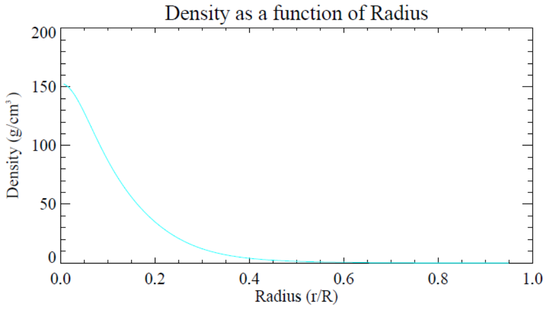
This gives $T_6 = 9.80$ $\quad \therefore T_c = \boxed{9.80\times10^6\ \mathrm{K}}$ [1] [7]

[Accept any valid numerical method to solve the equation, although several students may have a 'solve' button on their calculator or plot the graph and so not show much working. An evaluated value for ρ_c is not of use here as it would obscure its dependence on T, however if calculated then there is a mark in the next part that can be awarded here instead]

[The real value is 15.7 × 10^6 K, so our model has underestimated it by about a third – this is because we have assumed the density and the temperature are constant within the core, whilst in practice this is a poor approximation as shown in the graphs below. Interestingly, we could get closer to the real value by using our value of $\langle T_\odot \rangle$ and scaling it assuming that the plasma outside the core is much cooler than that within it, such that

$$T_c = \frac{M_\odot}{M_c}\langle T_\odot \rangle = \frac{1}{0.35}\times 4.73\times10^6 = 13.5\times10^6\ \mathrm{K}$$]





ii. Using this new central temperature, and considering R and the central number density of protons, n_p, estimate the typical amount of time a proton needs to wait to undergo fusion, giving your answer in years.

Evaluating the central density

$$\rho_c = \frac{p_c m_p}{k_B T\left(2X + \frac{3}{4}Y\right)}$$

$$= \frac{1.96\times10^{16}\times1.67\times10^{-27}}{1.38\times10^{-23}\times9.80\times10^{6}\times\left(2\times0.70+\frac{3}{4}\times0.30\right)} = 1.49\times10^{5}\ \mathrm{kg\ m^{-3}} \qquad [1]$$

[This mark may be awarded from working in part c. (i) if not explicitly present here. Since we have stated a constant density in the core, accept $\rho_c = \frac{0.35\,M_\odot}{\frac{4}{3}\pi\left(0.20\,R_\odot\right)^3} = 6.16\times10^{4}\ \mathrm{kg\ m^{-3}}$]

Using the evaluated expression for R with $T_6 = 9.80$

$$R = 6.55\times10^{-43}T_6^{-2/3}e^{-33.80T_6^{-1/3}} = 1.98\times10^{-50}\ \mathrm{m^3\ s^{-1}} \qquad [1]$$

The number density of protons in the centre of the core is

$$n_p = \frac{X\rho_c}{m_p} = \frac{0.70\times1.49\times10^{5}}{1.27\times10^{-27}} = 6.24\times10^{31}\ \mathrm{m^{-3}} \qquad [1]$$

Since $n_p R$ is the number of protons undergoing fusion every second, the reciprocal should give the expected time a proton has to wait to undergo fusion

$$\therefore t = \frac{1}{n_p R} = \frac{1}{6.24\times10^{31}\times1.98\times10^{-50}} = 8.10\times10^{17}\ \mathrm{s} = \boxed{2.57\times10^{10}\ \mathrm{yrs}} \qquad [1] \qquad [4]$$

[Final answer should be in years for the final mark. Watch out for students forgetting the factor of X in the expression for n_p and so losing the 3rd mark, but allow ecf for 4th mark (they get 1.80×10^{10} yrs). Using the alternative core density, they get 6.21×10^{10} yrs]

[The value for R really emphasises how little fusion is actually going on inside a cubic metre of the core of the Sun every second – it is simply the sheer size of the core that means enough pressure is produced to balance gravitational forces. Our value of ρ_c is very close to the real one of 1.52×10^5 kg m^{-3}, which is approximately 80 times denser than gold! Using the correct central temperature and pressure gives a main sequence lifetime of a little under 10^{10} years – the slowness of this reaction is the reason stars like the Sun stay on the main sequence for so long]

iii. Since $q \propto \tau^2 e^{-\tau}$ and $\tau \propto T^{-1/3}$, it can be approximated at a given temperature as $q \propto T^{\alpha}$, quantifying the sensitivity of the fusion reaction to temperature. By considering d(ln q)/d(ln T) give an expression for α as a function of τ and calculate it at your central temperature.

Inserting constants of proportionality into the expressions for q and τ

$$\tau = AT^{-1/3} \qquad \text{and} \qquad q = B\tau^2 e^{-\tau}$$

Differentiating both with respect to T (and using the product rule for q)

$$\frac{\mathrm{d}\tau}{\mathrm{d}T} = -\frac{1}{3}AT^{-4/3} = -\frac{\tau}{3T} \qquad [1]$$

$$\frac{\mathrm{d}q}{\mathrm{d}T} = 2B\tau e^{-\tau}\frac{\mathrm{d}\tau}{\mathrm{d}T} - B\tau^2 e^{-\tau}\frac{\mathrm{d}\tau}{\mathrm{d}T} \qquad [1]$$

Using the standard result that $\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x}$ then

$$\frac{\mathrm{d}(\ln q)}{\mathrm{d}(\ln T)} = \frac{T}{q}\frac{\mathrm{d}q}{\mathrm{d}T} = \frac{T}{B\tau^2 e^{-\tau}}(2B\tau e^{-\tau})\left(-\frac{\tau}{3T}\right) - \frac{T}{B\tau^2 e^{-\tau}}(B\tau^2 e^{-\tau})\left(-\frac{\tau}{3T}\right) \qquad [1]$$

$$= -\frac{2}{3} + \frac{\tau}{3} = \boxed{\frac{\tau - 2}{3}} \qquad \text{[must be simplified]} \qquad [1] \qquad [4]$$

[This is only one route – allow any valid method to get here, although the final expression must be as a function of τ and not a function of T (indicating the question was misread)]

By comparison with the evaluated expression for q and putting in the calculated value of T_c

$$\tau = 33.80\, T_6^{-1/3} = 33.80 \times 9.80^{-1/3} = 15.8 \qquad [1]$$

$$\therefore \alpha = \frac{15.79 - 2}{3} = \boxed{4.60} \qquad [1] \qquad [2]$$

[For the real central temperature $\alpha \approx 4$, showing the reasonable temperature dependence of the reaction – this is why very little fusion happens outside the core]

iv. The carbon-nitrogen-oxygen (CNO) cycle is an alternative pathway that becomes important at higher temperatures, where heavier elements catalyse the process of turning hydrogen into helium. There the rate limiting step is between nitrogen-14 and hydrogen-1. Compare the temperature dependence of the CNO cycle to the p-p chain at the Sun's central temperature.

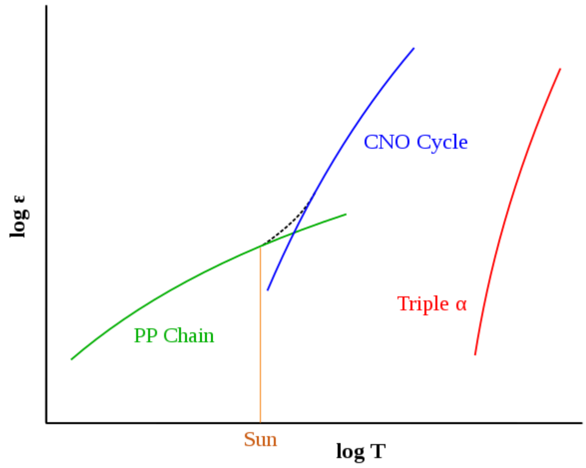
For the reaction between nitrogen-14 and hydrogen-1

$$\mu_r = \frac{A_i A_j}{A_i + A_j} = \frac{14 \times 1}{14 + 1} = \frac{14}{15} \qquad [1]$$

$$\tau = 42.59\left[Z_i^2 Z_j^2 \mu_r T_6^{-1}\right]^{1/3} = 42.59\left[7^2 \times 1^2 \times \frac{14}{15} \times 9.80^{-1}\right]^{1/3} = 71.2 \qquad [1]$$

$$\therefore \alpha = \frac{71.2 - 2}{3} = \boxed{23.06} \qquad [1] \qquad [3]$$

This is a much steeper temperature dependence than the p-p chain [1] [1]



[This value of α is why the CNO cycle dominates at higher temperatures yet is almost irrelevant in the Sun. At even higher temperatures the triple alpha process (a way of fusing helium nuclei into carbon) dominates with an even steeper temperature dependence of $\alpha \approx 41$ – as you use nucleosynthesis to move along the periodic table the value of α tends to increase. This is the reason why the regions of different nucleosynthesis occur in well-defined shells in the star after it leaves the main sequence.]

Q3 – James Webb Space Telescope [35 marks]

a. The telescope will spend its expected 10-20 year mission in a halo orbit about the second Lagrangian point, L2 (see Figure 5). This is one of five special points in the Sun-Earth system where the gravitational forces from the two bodies provide the centripetal force required to have a (small mass) object there have an orbital period identical to the Earth. At the L2 point, this means it orbits quicker than you would expect for an object that distance from the Sun.

i. Taking 1 year as 365.25 days and 1 au as 1.496×10^{11} m, using numerical methods show that the distance between the Earth and L2 is ~ 1.5×10^{6} km. Give your answer to 4 s.f.

At the L2 point the gravitational forces from the Sun and Earth balance the centripetal force

$F_{G,\odot} + F_{G,E} = F_{centripetal}$ [can be implied by 2nd mark] [1]

Defining x to be the distance between the Earth and L2 and m to be mass of the probe

$$\frac{GM_{\odot}m}{(1\ \text{au}+x)^2} + \frac{GM_E m}{x^2} = m(1\ \text{au}+x)\frac{4\pi^2}{(1\ \text{year})^2} \quad [1]$$

$$\therefore GM_{\odot}x^2 + GM_E(1\ \text{au}+x)^2 - x^2(1\ \text{au}+x)^3\frac{4\pi^2}{(1\ \text{year})^2} = 0 \quad [1]$$

Using any reasonable root-finding numerical method with $x_0 = 1.5 \times 10^9$ m [1]

$\boxed{x = 1.502 \times 10^9\ \text{m}}$ (so roughly 1.5×10^6 km, as expected) [1] [5]

[The 3rd mark is for creating a suitable quintic and setting it equal to zero. In expanded form it is $-3.96417 \times 10^{-14}x^5 - 0.0177912\,x^4 - 2.66156 \times 10^9 x^3 + 1.0278 \times 10^{16}x^2 + 1.19141 \times 10^{26}x + 8.91176 \times 10^{36} = 0$. For the 4th mark the Newton-Raphson method works well with the expanded form, although, since the starting value is so close to the root, interval bisection or decimal search methods are much quicker to do and don't require the expanded form – **this is the recommended approach for this sort of problem**. Since the required root is the only non-complex root of the quintic, even methods that are slow to converge will get there. The final answer must be 4 s.f. for the final mark.]

[Several other spacecraft has been sent to L2, such as the WMAP and Planck probes which looked at the cosmic microwave background. L2 is about 4 times further away than the Moon and is roughly a million miles away (which brings context to the common saying!)]

ii. The JWST is on an orbit that will take it to within 200 000 km of L2, where it will then do a final large burn of the rockets to insert it into the halo orbit around L2. Assuming it is on a simple elliptical transfer orbit ignoring the influence of the Sun, and had a perigee at an altitude of 2100 km above the surface of the Earth, how long will it take JWST to get to the L2 orbital insertion phase of its mission? Give your answer in days.

Finding the semi-major axis of the elliptical orbit

$$a = \frac{1}{2}(r_{perigee} + r_{apogee})$$
$$= \frac{1}{2}\left((2100 + 6370) \times 10^3 + (1.502 - 0.2) \times 10^9\right) = 6.55 \times 10^8\ \text{m} \quad [1]$$

Using Kepler's 3rd Law

$$T^2 = \frac{4\pi^2}{GM_E}a^3 \qquad \therefore T = \sqrt{\frac{4\pi^2}{6.67\times10^{-11}\times5.97\times10^{24}}(6.55 \times 10^8)^3}$$
$$= 5.28 \times 10^6\ \text{s}$$ [allow ecf with their a] [1]

We want only half the period to get the time to go from apogee to perigee

$\frac{1}{2}T = 2.64 \times 10^6\ \text{s} = \boxed{30.6\ \text{days}}$ [must be in days for final mark] [1] [3]

[The question intended for the probe to be 200 000 km shy of L2, however it is ambiguous from the phrasing so accept if the student treats it as 200 000 km beyond L2 (getting $a = 8.55 \times 10^8$ m, $T = 7.88 \times 10^6$ s and so a final mission time of $\boxed{45.6\ \text{days}}$)]

[The expected time to get to L2 was about a month, so our elliptical transfer approximation has not done too badly – the real orbital shape is much more complex. Due to the time taken to get there, it would be impossible for any human mission to be launched to fix any faults with the telescope (as needed to happen with the Hubble Space Telescope)]

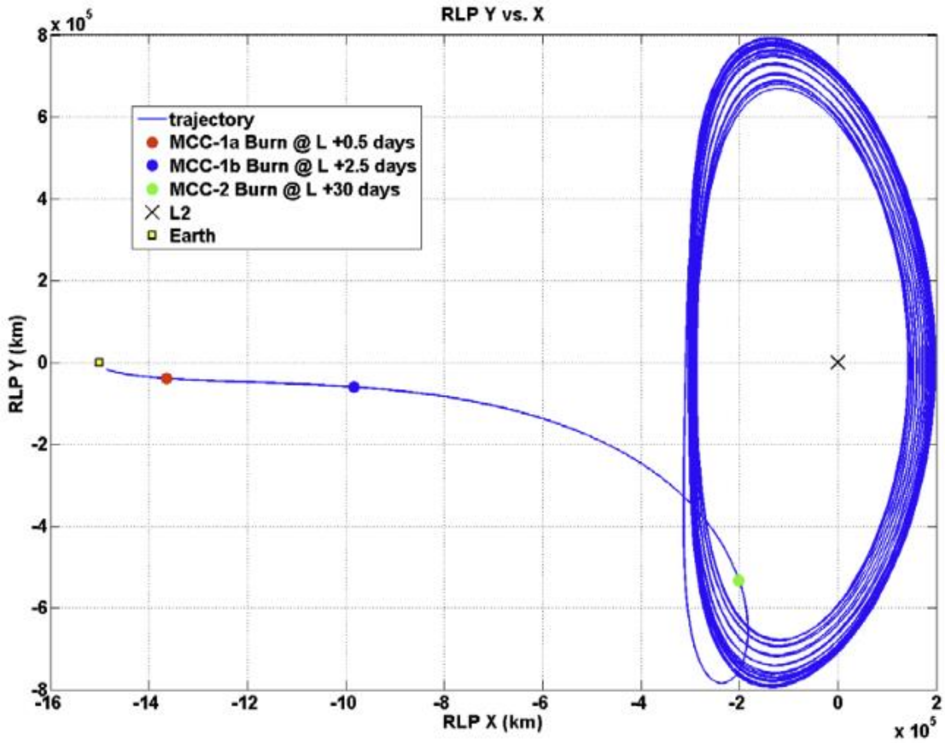


Figure 2. Representative Trajectory with JWST MCC maneuvers

b. To achieve suitable sampling, an image will be considered diffraction limited when it has ≥ 2 pixels per θ_{FWHM}. The diameter of the JWST primary mirror is 6.5 m, however since it is composed of hexagons and hexagonal in shape, it is not straightforward to work out the equivalent circular mirror diameter. To a good approximation it can be taken to be 6.0 m.

i. Given $\theta_{FWHM} = \alpha\lambda/D$, find α for I_{circ}, giving your answer to 3 s.f.

Substituting in the given expansion of $J_1(x)$

$$I_{circ} = I_0\left(\frac{2J_1(x)}{x}\right)^2 = I_0\left(1-\frac{x^2}{8}+\frac{x^4}{192}\right)^2 \qquad [1]$$

At FWHM, $I_{circ} = \frac{1}{2}I_0$

$$\therefore \frac{1}{2} = \left(1-\frac{x^2}{8}+\frac{x^4}{192}\right)^2 \therefore \frac{1}{192}x^4 - \frac{1}{8}x^2 + \left(1-\frac{1}{\sqrt{2}}\right) = 0 \qquad [1]$$

Any reasonable method to solve this quartic [1]

[Expected approach is to use the substitution $y = x^2$ and solve the resulting quadratic to give $y = 21.37$ or $y = 2.632$]

$x = \pm 4.623$ or $x = \pm 1.622$ [Don't penalise if missing $\pm$] [1]

Considering only the positive roots of the quartic , and since $\frac{1}{2}\alpha = \frac{x}{\pi}$

$\frac{1}{2}\alpha = 1.471$ or $\frac{1}{2}\alpha = 0.516$ [1]

$\therefore \alpha = 2.943$ or $\alpha = 1.033$ [1]

From Fig 6 it is clear that $\theta_{FWHM} < \theta_{min} \therefore \alpha < 1.22 \therefore \boxed{\alpha = 1.03}$ [1] [7]

[Assuming $\alpha = x/\pi$ means only lose 6th mark – allow ecf on correct root choice for 7th. Must be 3 s.f. for final mark. Expanding the bracket on the second line and only going up to the x^4 term leads to α = 1.07 – this approach only loses the second mark and has ecf for the rest]

ii. Hence, determine which of the three imaging instruments is diffraction limited for the greatest fraction of its wavelength range.

At the critical wavelength, $\theta_{FWHM} = 2 \times \text{plate scale (in rad)}$ [1]

$$\theta_{FWHM} = \frac{1.03\lambda}{D} \therefore \lambda = \frac{D\theta_{FWHM}}{1.03} \therefore \lambda_{crit} = \frac{2D \times \text{plate scale}}{1.03}$$

For each instrument

$$\lambda_{crit,NIRCam-SW} = \frac{2 \times 6.0 \times \left(0.031 \times \frac{\pi}{3600 \times 180}\right)}{1.03} = 1.75\ \mu\text{m} \quad [1]$$

$$\lambda_{crit,NIRCam-LW} = 3.66\ \mu\text{m} \quad [1]$$

$$\lambda_{crit,MIRI} = 6.20\ \mu\text{m} \quad [1]$$

Images will be diffraction limited when $\theta_{FWHM} \geq 2 \times \text{plate scale} \therefore \lambda \geq \lambda_{crit}$ [1]

Diffraction limited wavelength range for each instrument

NIRCam-SW	$1.75 - 2.3\ \mu\text{m}$	(33% of range)	
NIRCam-LW	$3.66 - 5.0\ \mu\text{m}$	(51% of range)	
MIRI	$6.20 - 25.5\ \mu\text{m}$	(97% of range)	$\therefore$ MIRI [1] [6]

[A bald statement of MIRI without working receives only the final mark. The first mark is for some understanding of how to use the given plate scale (either as a statement or in a formula). The percentage of each range is not required for the final mark. No ecf on final mark if 5th mark is not achieved (e.g. student incorrectly looks at $\lambda \leq \lambda_{crit}$). Full ecf on their value of α]

[This is why JWST is not quite considered to be the direct successor of the Hubble Space Telescope (HST), as in the orange / red regions of the visible that it can image in it will not be diffraction limited (unlike the Hubble Space Telescope which was diffraction limited throughout almost the whole visible range) – its specialism is much more in the infrared]

c. Computer models suggest the first galaxies formed around $z \sim 10 - 20$. One of the best ways to look for high-redshift galaxies is to try and detect the emission from the Lyman alpha (Lyα) emission line at λ_{emit} = 121.6 nm as it is a relatively bright line. Some of the brightest galaxies in that initial era of galaxy formation would have an absolute magnitude of $\mathcal{M} \sim 20$. In this question, you are given that $\Omega_{0,m} = 0.3$, $\Omega_{0,\Lambda} = 0.7$, $\Omega_{0,r} = 0$ and $H_0 = 70$ km s^{-1} Mpc^{-1}.

i. Calculate the redshift at which the Lyα line is detected in the centre of the F200W filter.

Given that Lyα = 121.6 nm and the F200W filter is centred at 1.989 μm

$$z = \frac{\lambda_{obs} - \lambda_{emit}}{\lambda_{emit}} = \frac{1989 - 121.6}{121.6} = \boxed{15.4} \quad [1] \quad [1]$$

ii. How long after the Big Bang does this correspond to? Give you answer in years.

Calculating the Hubble time (after converting the units of H_0) [Accept formula sheet H_0]

$$t_{H_0} = \frac{1}{H_0} = \frac{3.09 \times 10^{16} \times 10^6}{70 \times 10^3} = 4.41 \times 10^{17}\ \text{s} \quad (= 1.40 \times 10^{10}\ \text{years}) \quad [1]$$

Substituting in the values of z, t_{H_0}, $\Omega_{0,m} = 0.3$ and $\Omega_{0,\Lambda} = 0.7$ into the given formula

$$t = t_{H_0} \frac{2}{3\Omega_{0,\Lambda}^{1/2}} \ln\left[\left(\frac{\Omega_{0,\Lambda}}{\Omega_{0,m}}\right)^{1/2} (1+z)^{-3/2} + \left(\frac{\Omega_{0,\Lambda}}{\Omega_{0,m}(1+z)^3} + 1\right)^{1/2}\right]$$

$$= 1.40 \times 10^{10} \times \frac{2}{3 \times \sqrt{0.7}} \ln\left[\left(\frac{0.7}{0.3}\right)^{1/2} (1+15.4)^{-3/2} + \left(\frac{0.7}{0.3(1+15.4)^3} + 1\right)^{1/2}\right] \quad [1]$$

$$= \boxed{2.58 \times 10^8\ \text{years}} \quad \text{[must be in years]} \quad [1] \quad [3]$$

[This shows that JWST is probing a period in the Universe's history when it is less than 2% of its current age!]

iii. Calculate the luminosity distance to the galaxy and hence its apparent magnitude. Assume all emitted flux is picked up by the telescope.

Calculating the Hubble distance

$$D_{H_0} = ct_{H_0}$$
$$= 3.00 \times 10^8 \times 4.41 \times 10^{17} = 1.32 \times 10^{26}\ \text{m}\ \ (= 4286\ \text{Mpc}) \qquad [1]$$

Finding the scale factor corresponding to $z = 15.4$

$$a = (1+z)^{-1} = (1+15.4)^{-1} = 0.0611 \qquad [1]$$

This means the integral that needs to be done is

$$\int_{0.0611}^{1} (0.3a + 0.7a^4)^{-1/2}\,\text{d}a \qquad [1]$$

[This mark could also be awarded for correct substitution into the integral with z]

Any reasonable method to numerically integrate this expression [1]

$$\text{Integral} = 2.40225\ldots \qquad [1]$$

[Trapezium rule with 10 steps gives 2.34, Simpson's rule with 10 steps gives 2.41, most calculator's inbuilt functions will give 2.40 – in general accept 2.40 ± 0.06]

Hence the luminosity distance is

$$D_L = (1+z_i)D_{H_0} \times \text{integral}$$
$$= (1+15.4) \times 4.286 \times 2.402 = \boxed{168\ \text{Gpc}} \qquad [\text{Accept} \pm 5\ \text{Gpc}] \qquad [1] \qquad [6]$$

Consequently, the apparent magnitude is

$$m - \mathcal{M} = 5\log\left(\frac{D_L}{10}\right) \quad \therefore m = 5\log\left(\frac{D_L}{10}\right) + \mathcal{M}$$
$$= 5\log\left(\frac{168\times10^9}{10}\right) - 20 = \boxed{31.1} \qquad [1] \qquad [1]$$

[This is as faint as any object detected by HST in the Extreme Deep Field (XDF). In practice, the filter only picks up part of the emitted spectrum and so we need to take into account broadening of the Lyα line due to the redshift and an assumed spectral slope to convert between the detected and emitted flux – this is a much more involved calculation. Supernovae from the deaths of the very first stars would have a similar apparent magnitude so JWST will be able to investigate the physics of the earliest stellar populations too]

iv. If the minimum flux detectable decreases proportionally to $t_{exp}^{1/2}$ where t_{exp} is the length of the exposure, estimate the minimum exposure time necessary for JWST to image this galaxy with S/N = 10. Give your answer in hours.

Calculating the ratio of the flux from the galaxy to the given limit after 10^4 s

$$\frac{b_{lim}}{b_{gal}} = 10^{-0.4(m_{lim}-m_{gal})} = 10^{-0.4(29.0-31.1)} = 7.12 \qquad [1]$$

We are told $b \propto t_{exp}^{-1/2}$

$$\therefore \frac{b_{lim}}{b_{gal}} = \left(\frac{10^4}{t_{exp}}\right)^{-1/2} \quad \therefore t_{exp} = 10^4 \times \left(\frac{b_{lim}}{b_{gal}}\right)^2 \qquad [1]$$
$$= 10^4 \times 7.12^2 = 5.07 \times 10^5\ \text{s} = \boxed{141\ \text{hrs}} \qquad [1] \qquad [3]$$

[The 1st mark can also be given for $b_{gal} = 1.28$ nJy. Needs to be in hours for final mark]

[In practice, S/N = 5 (as used by the HST XDF) would be sufficient for most discovery purposes which would be about half the time. JWST was designed to be at its most sensitive at 2 μm where it also has the best angular resolution of the diffraction limited range of any of its cameras – this should give an amazing opportunity to explore the high redshift Universe in unprecedented detail in reasonable timescales]