BAAO 2022/23 Solutions and Marking Guidelines

Note for markers:

- Answers to two or three significant figures are generally acceptable. The solution may give more in order to make the calculation clear. Units should be present on final answers when appropriate.
- There are multiple ways to solve some of the questions; please accept all good solutions that arrive at the correct answer. Students getting the answer in a $\boxed{\text{box}}$ will get all the marks available for that calculation / part of the question (as indicated in red), so long as there are no unphysical / nonsensical steps or assumptions made (students may not explicitly calculate the intermediate stages and should not be penalised for this so long as their argument is clear).

Q1 – Solar Analemma [35 marks]

a. Although α is really an angle in radians (where 2π radians = 360°), it is normally more useful to convert it into time units (essentially the time since the Sun was due South, or the time until the Sun reaches due South). Taking the mean solar day to be exactly 24 hours:

i. Convert the amplitude of α_{tilt} and α_{ecc} for the Earth into minutes.

To convert from radians to minutes of time need to multiply by a factor of:

$$\frac{24\times60}{2\pi}=\frac{720}{\pi} \qquad [1]$$

Putting the numbers into the given formula:

$$\alpha_{tilt} = \tan^2\left(\frac{23.44°}{2}\right) = 0.0430 \text{ rad} = \boxed{9.86 \text{ minutes}} \ (= 9 \text{ m } 51.8 \text{ s}) \qquad [1]$$

$$\alpha_{ecc} = 2\times0.0167 = 0.0334 \text{ rad} = \boxed{7.65 \text{ minutes}} \ (= 7 \text{ m } 39.3 \text{ s}) \qquad [1] \qquad [3]$$

[If they only calculate both α in radians, award 1 mark. If they confuse the needed conversion with arcminutes (giving α_{tilt} = 148′ and α_{ecc} = 115′), lose first mark but give ecf. Conversion done correctly to just one α means max 2 marks. **Allow ecf for later parts**.]

ii. Determine equations for δ (in degrees) and tilt and ecc (both in minutes) as a function of the day of the year, n. Take n = 1 to be 1st January and n = 365 to be 31st December.

Considering the number of days between the winter solstice (21st Dec) and 1st Jan, which is a minimum, then

$$\boxed{\delta = -23.44°\cos\left(2\pi\frac{n+10}{365}\right)} \qquad [1] \qquad [1]$$

Since the winter solstice is a zero point, and α_{tilt} is negative after it, with double the frequency of δ:

$$\boxed{\alpha_{tilt} = -9.86\sin\left(4\pi\frac{n+10}{365}\right)} \qquad [1] \qquad [1]$$

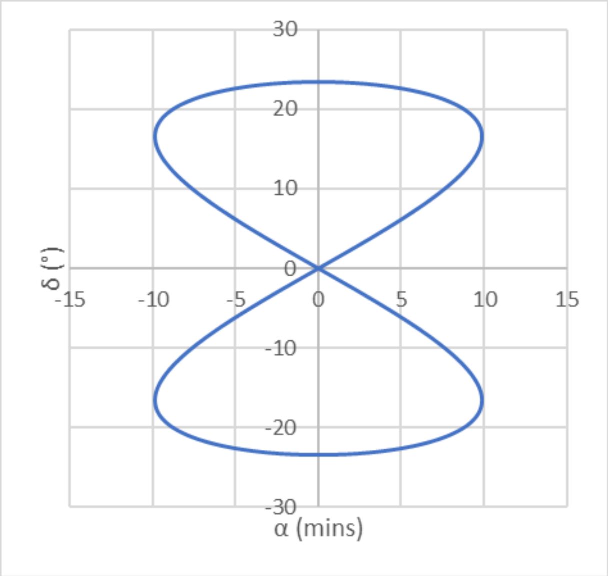
Since perihelion on 4th Jan is a zero point and α_{ecc} is negative after it, with the same frequency as δ

$$\boxed{\alpha_{ecc} = -7.65\sin\left(2\pi\frac{n-4}{365}\right)} \qquad [1] \qquad [1]$$

[**Allow any equivalent expressions** in sin or cos, or using degrees inside the trigonometric functions. In these equations we are assuming midday on 1st Jan has n = 1 (so n = 0 & n = 365 at midday on 31st Dec) – do not penalise students for choosing a different zero point (e.g. midnight on 1st Jan) and hence getting slightly different additive terms to their n (allow ±1)]

iii. Sketch the analemma you would see if the Earth's orbit was circular (i.e. e = 0) but with the current value of ε, with α on the x-axis (in minutes) and δ on the y-axis.

Correct shape, with rotational and reflectional symmetry and passing through (0,0) [1]
α axis labelled and extrema labelled as -9.86 mins and 9.86 mins (allow ecf from a)i)) [1]
δ axis labelled and extrema labelled as -23.44° and 23.44° [1]



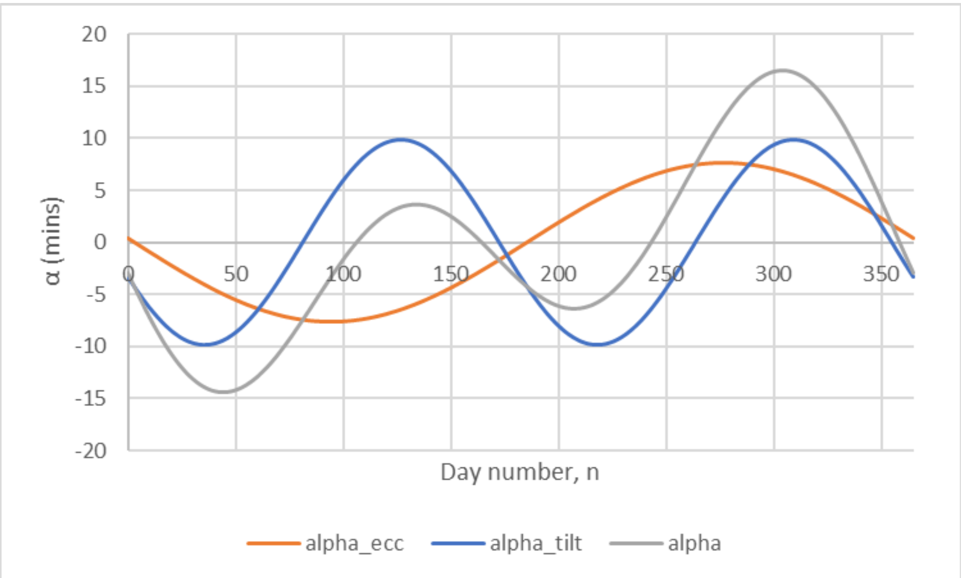
[3]

b. Using your equations derived in the previous part, sketch on GRAPH PAPER on the same set of axes:

i. α_{ecc} (vertical axis) in minutes against n (horizontal axis).

ii. α_{tilt} in minutes against n.

iii. Hence, carry out the superposition of those two waves to get α against n. [Note: a clear indication of the general shape will do - no need to spend very long on making this perfect]



Part i) is the orange line [1] [1]
Part ii) is the blue line [1] [1]
Part iii) is the grey line
Has two maxima / minima (one bigger than the other) in ≈ the right place [1]
Largest maxima is larger amplitude than the lowest minima [1] [2]
[Allow ecf from earlier parts. Lose 0.5 marks for missing axes labels]

c. Using your graph to guide you to the relevant point of the year, but using your precise algebraic expressions (rather than reading off the graph):

i. What are the dates and values of the maximum and minimum values of α?

We can use the differential of α to find turning points, and then we know from the graph we want the ones near n ≈ 50 and n ≈ 300

$$\alpha = \alpha_{tilt} + \alpha_{ecc} = -9.86 \sin\left(4\pi \frac{n+10}{365}\right) - 7.65 \sin\left(2\pi \frac{n-4}{365}\right)$$

$$\therefore \frac{d\alpha}{dn} = -9.86 \times \frac{4\pi}{365} \cos\left(4\pi \frac{n+10}{365}\right) - 7.65 \times \frac{2\pi}{365} \cos\left(2\pi \frac{n-4}{365}\right) \quad [1]$$

Looking where dα/dn = 0 (using any suitable method e.g. binary search) we find turning points in α at

$n = 44, 134, 207, 304$ [0.5 marks for each] [2]

We want the first and last values:

$n = 44 \therefore \text{date} = \boxed{13^{\text{th}}\ \text{Feb}}$ and $\alpha = \boxed{-14.31\ \text{mins}}$ [1]

$n = 304 \therefore \text{date} = \boxed{31^{\text{st}}\ \text{Oct}}$ and $\alpha = \boxed{+16.57\ \text{mins}}$ [1] [5]

[Do not penalise non-integer n. Allow n and dates consistent with their counting system (so may be different by ±1). For the final two marks it is 0.5 marks for a correct date and 0.5 marks for a correct α]

[Our equation for alpha is known as the Equation of Time, and positive values mean that the Sun is ahead of the clock (so to the right of due south in the photo). Despite our approximations for the two components of α, we are relatively close to the real extrema: -14.25 mins on 11th Feb and +16.42 mins on 3rd Nov]

ii. What are the dates when the Sun is due South at midday?

These correspond to when α = 0 [1]
Using any suitable method (e.g. binary search)

$n = 106, 164, 243, 359$ [0.5 marks for each] [2]

These correspond to the following dates:

$\boxed{16^{\text{th}}\ \text{Apr}}, \boxed{13^{\text{th}}\ \text{Jun}}, \boxed{31^{\text{st}}\ \text{Aug}}$ and $\boxed{25^{\text{th}}\ \text{Dec}}$ [0.5 marks for each] [2] [5]

[Do not penalise non-integer n. Allow n and dates consistent with their counting system (so may be different by ±1)]

[The real values are 15th Apr, 13th Jun, 1st Sep, and 25th Dec, so our approximations have done very well]

iii. What are the dates when the Sun passes between the large and small loops in the figure-of-eight (i.e. the crossover point)? [Note: α ≠ 0 at that point, but is small]

We are told that α is small and can see from the photo that it is not near the solstices, so must be near the zero points in α far from the solstices i.e. 16th Apr and 31st Aug [1]

On 16th Apr δ is increasing, and on 31st Aug δ is decreasing. The know that δ at the crossover is equal on both dates (due to the equation for δ being double valued in the range of n) and between their two values of δ therefore the relevant dates must be **earlier** [1]

Using a suitable method (e.g. starting a binary search with these points as initial values)

$$n = 106, \alpha = -0.10 \text{ mins}, \delta = 9.69° \qquad n = 243, \alpha = -0.14 \text{ mins}, \delta = 8.20°$$
$$n = 105, \alpha = -0.35 \text{ mins}, \delta = 9.32° \qquad n = 242, \alpha = -0.47 \text{ mins}, \delta = 8.57°$$
$$n = 104, \alpha = -0.61 \text{ mins}, \delta = 8.95° \qquad n = 241, \alpha = -0.79 \text{ mins}, \delta = 8.95°$$
$$n = 103, \alpha = -0.87 \text{ mins}, \delta = 8.57° \qquad n = 240, \alpha = -1.10 \text{ mins}, \delta = 9.32°$$

[2]

$$\therefore n = 104 \text{ so date} = \boxed{14^{\text{th}} \text{ Apr}} \text{ and n} = 241 \text{ so date} = \boxed{29^{\text{th}} \text{ Aug}}$$ [1] [5]

[A wide variety of numerical methods might be employed here, although binary search close to the roots of α is probably the quickest. Allow ecf considerations as in earlier steps. Give 0.5 marks for each correct date]

[The real dates are 12th Apr and 29th Aug, so we have again done rather well]

d. Considering data from Figure 2, determine for Earth 2.0:

i. The axial tilt, ε.

The extrema in δ will be $\pm\varepsilon$ so just reading off the graph $\boxed{\varepsilon = 15°}$ [1] [1]

ii. The month of the vernal equinox.

At the vernal equinox, δ = 0 and is increasing, so it must be $\boxed{\text{November}}$ [1] [1]

[Allow 0.5 marks for May, if it is clear it was chosen due to requiring δ = 0]

iii. The eccentricity of the orbit, e.

With the information in the previous two parts, we can create an equation for α

The component due to eccentricity has the same general form

$$\alpha_{ecc} = -2e \sin\left(2\pi \frac{n-4}{365}\right)$$ [0.5]

Using the earlier equations for α_{tilt}

$$\alpha_{tilt} = \tan^2\left(\frac{15°}{2}\right) = 0.0173 \text{ rad} = 3.97 \text{ minutes}$$ [0.5]

Since the vernal equinox is now 21st November (30 days earlier than the winter solstice used to be), our sinusoidal equation can be modified to

$$\alpha_{tilt} = 3.97 \sin\left(4\pi \frac{n+40}{365}\right)$$ [Accept equivalent forms] [0.5]

Putting in data for one extremum (e.g. -26 mins 47 secs = -26.783 mins) and knowing it is a turning point (so $\frac{d\alpha}{dn} = 0$) gives us two equations:

$$-2e \sin\left(2\pi \frac{n_1-4}{365}\right) + 3.97 \sin\left(4\pi \frac{n_1+40}{365}\right) = -26.783 \qquad ①$$

$$-2e \times \frac{2\pi}{365} \cos\left(2\pi \frac{n_1-4}{365}\right) + 3.97 \times \frac{4\pi}{365} \cos\left(4\pi \frac{n_1+40}{365}\right) = 0 \qquad ②$$ [0.5]

Combining ① and ② to eliminate e, gives the equation

$$\tan\left(2\pi\frac{n_1-4}{365}\right)-\frac{1}{2}\tan\left(4\pi\frac{n_1+40}{365}\right)-\frac{26.783}{3.97\times2\cos\left(4\pi\frac{n_1+40}{365}\right)}=0 \qquad [1]$$

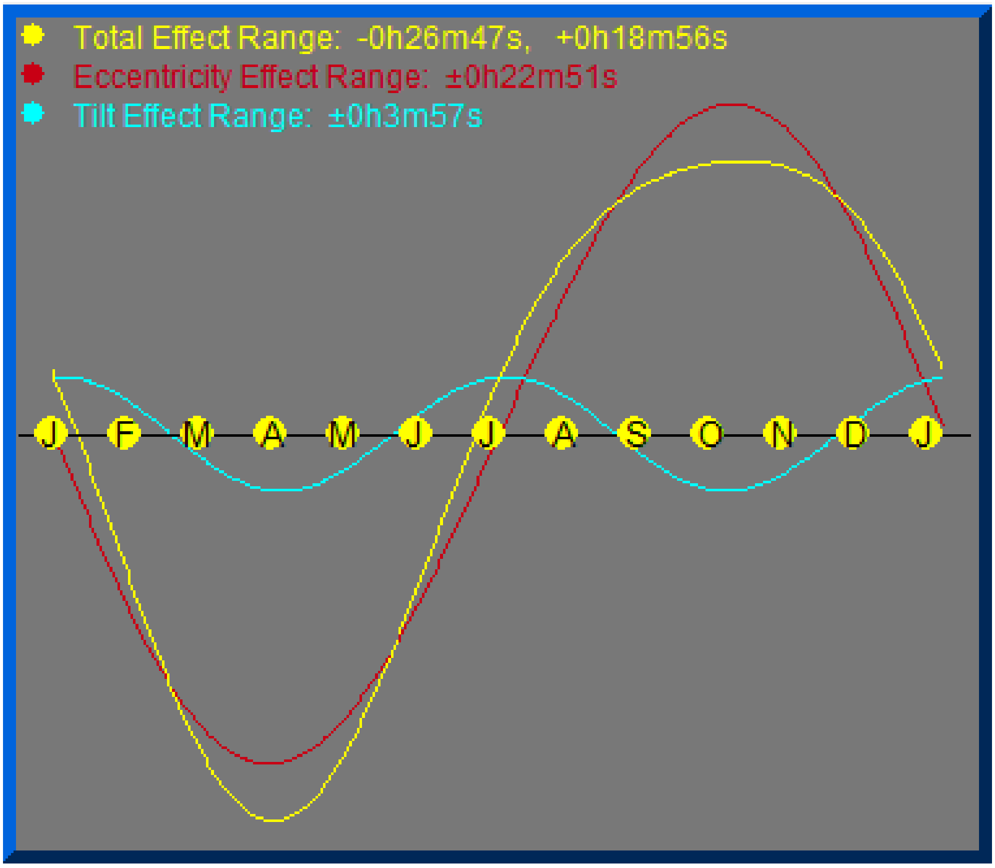
Using an appropriate numerical method to solve it for n_1, the only valid values in the allowed region are

$$n_1 = 96 \text{ or } 278 \qquad [1]$$

Recognising that the relevant extremum is around April, n_1 = 96 is the correct one to choose. Putting this into either ① or ② leads to $\boxed{e = 0.0498}$ [1] [5]

[Using the other extremum gets $n_2 = 92$ or 274 and so recognising the latter is the correct one to use gives $\boxed{e = 0.0499}$. The method outlined here works in general, however in this specific case an alternative (much shorter) method will work; α_{tilt} has minima at n = 97 and 279, which almost exactly coincide with the turning points of α_{ecc} at n = 95 (min) and 278 (max). Consequently, by considering how the graph of α_{ecc}, α_{tilt} and α would look, it is clear that $-2e - 3m57 \approx -26m47$ and $2e - 3m57 \approx 18m56$. Taking the average of the modulus of the extrema in α, and assuming that is 2e gives $\boxed{e = 0.0499}$. Students that use this approach to take advantage of the greatly simplifying phase relationship between α_{ecc} and α_{tilt} receive full credit (same first 1.5 marks, 1 mark for turning points, 0.5 marks for recognising the consequences of the tuning points being close together, 1 mark for setting up equations, 1 mark for final answer). Those whose working implies they did not recognise this was a special case (and hence suggest this method would always work) get 1 mark only for a correct final answer]

The relevant graph of superposition is below, emphasising why in this case the simple method would work due to the alignment of the minima of α_{tilt} with the max / min of α_{ecc}.



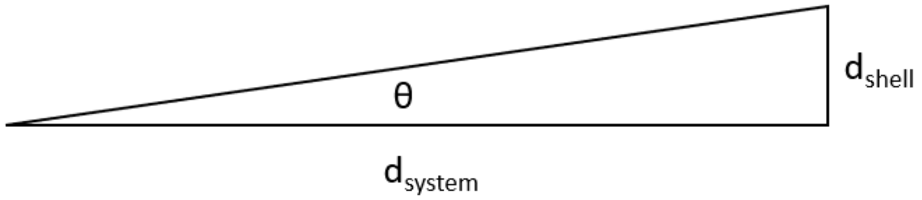
Q2 – Wolf-Rayet Dust Shells [40 marks]

a. Take the distance to the system to be 1.64 kpc.

i. By taking measurements from Figure 3, show that the average radial expansion velocity between shells 2 to 17 is about 2600 km s^{-1}.

Measuring from the graph, the average spacing between shells is 2.67” [1]

[Expect to see candidates realise that 40” covers the 15 intervals from shell 2 to 17, and hence the average separation is 40”/15 = 2.67”. Allow ±0.03” and ecf on all subsequent parts of the question accordingly]



Using the small angle approximation (with the angle converted into radians), the distance between shells implied by their average angular spacing is

$$d_{shell} = d_{system} \tan\theta \approx d_{system}\theta = 1.64 \times \left(\frac{2.67}{3600} \times \frac{2\pi}{360}\right)$$
$$= 2.12 \times 10^{-5}\ \text{kpc} = 6.56 \times 10^{11}\ \text{km} \quad [1]$$

Given the period is 2895 days and converting it to seconds,

$$v_{shell} = \frac{d_{shell}}{T} = \frac{6.56\times10^{11}}{2895\times24\times60\times60} = 2622.6 \ldots = \boxed{2620\ \text{km s}^{-1}} \quad [1] \quad [3]$$

[Given it is a ‘show that’, it must have ≥ 3 s.f. in final answer]

ii. Shell 1 was observed by JWST on 27th July 2022 to be 1.63” away from the central stars. It was formed during the last periastron passage of the O star, which (as viewed from Earth) took place in December 2016. Taking light travel time into account, in what year was the periastron passage responsible for shell 17?

From shell 1 to shell 17 is 16 intervals, so in years

$$\Delta t = 16 \times \left(\frac{2895}{365}\right) = 126.9\ \text{yrs} \quad [1]$$

[Accept use of 365.25 days in a year]

The light travel time from the system is

$$t_{system} = \frac{d_{system}}{c} = \frac{1.24\times1000\times3.09\times10^{16}}{3.00\times10^{8}} = 1.69 \times 10^{11}\ \text{s} = 5356.4\ \text{yrs} \quad [1]$$

Given that December 2016 is approximately decimalised as 2016.9, and that 1 BC goes to 1 AD with no year zero, then the year it was created is

$$2016.9 - (126.9 + 5356.4) - 1 = -3467.4 = (\text{July})\ \boxed{3467\ \text{BC}} \quad [1] \quad [3]$$

[Accept use of 2016 instead of 2016.9, and not skipping year 0, giving an allowance of ±2 years on the final answer]

b. In the Han et al. (2022) model they use r_{inner} = 50 au, r_{outer} = 220 au, and assume spherical dust grains with a radius of 40 nm and density 1.6 g cm^{-3}, as well as a dust-to-gas ratio of 0.019. Take the apparent magnitude of the system (after correcting for substantial extinction by the dust shells and the general ISM) as m = 0.426, and you are given the absolute magnitude of the Sun is $\mathcal{M}_\odot$ = 4.74.

i. Show that a_{max} is about 940 km s^{-1} yr^{-1}.

First we can find the absolute magnitude of the system

$$\mathcal{M} = m - 5\log\left(\frac{d_{system}}{10}\right) = 0.426 - 5\log\left(\frac{1.64\times10^3}{10}\right) = -10.65 \qquad [1]$$

By comparison with the Sun we can turn this into a luminosity

$$L = 10^{\frac{\mathcal{M}-\mathcal{M}_\odot}{-2.5}} L_\odot = 10^{\frac{-10.65-4.74}{-2.5}} L_\odot = 1.43\times10^6\, L_\odot = 5.50\times10^{32}\ \text{W} \quad [1]$$

From Newton's 2nd Law

$$a_{max} = \frac{F_{rad}}{m_{tot}} = \frac{F_{rad}}{m_{dust}} \times \frac{m_{dust}}{m_{tot}} \approx \frac{F_{rad}}{m_{dust}} \times \frac{m_{dust}}{m_{gas}} = \frac{\sigma_g}{m_{dust}} \frac{L_{bin}}{4\pi r_{outer}^2 c} \times \frac{m_{dust}}{m_{gas}}$$

$$= \frac{\pi r_{dust}^2}{\rho_{dust}\times\frac{4}{3}\pi r_{dust}^3} \frac{L_{bin}}{4\pi r_{outer}^2 c} \times \frac{m_{dust}}{m_{gas}} = \frac{3L_{bin}}{16\rho_{dust} r_{dust}\pi r_{outer}^2 c} \times \frac{m_{dust}}{m_{gas}} \qquad [1]$$

$$\therefore a_{max} = \frac{3\times5.50\times10^{32}}{16\times1600\times40\times10^{-9}\times\pi\times(220\times1.50\times10^{11})^2\times3.00\times10^8} \times 0.019$$

$$= 2.99\times10^{-2}\ \text{m s}^{-2} \qquad [1]$$

$$= \boxed{941\ \text{km s}^{-1}\ \text{yr}^{-1}} \qquad [1] \quad [5]$$

[Given it is a 'show that', it must have ≥ 3 s.f. in final answer. Final mark is for correct unit conversion. Give full credit for the use of the exact corrective factor $\frac{m_{dust}}{m_{tot}} = \frac{m_{dust}}{m_{dust}+m_{gas}} = \frac{1}{1+m_{gas}/m_{dust}} = 0.01865$, leading to a_{max} = 2.93 × 10^{-2} m s^{-2} = $\boxed{924\ \text{km s}^{-1}\ \text{yr}^{-1}}$. Forgetting about the use of the dust-to-gas ratio gives a_{max} = 1.57 m s^{-2} = 49554 km s^{-1} yr^{-1} and scores 4 marks max. If no attempt at the algebraic simplification, the third mark can be gained for either a correct m_{dust} (= 4.29 × 10^{-19} kg) **or** σ_g (= 5.03 × 10^{-15} m^2)]

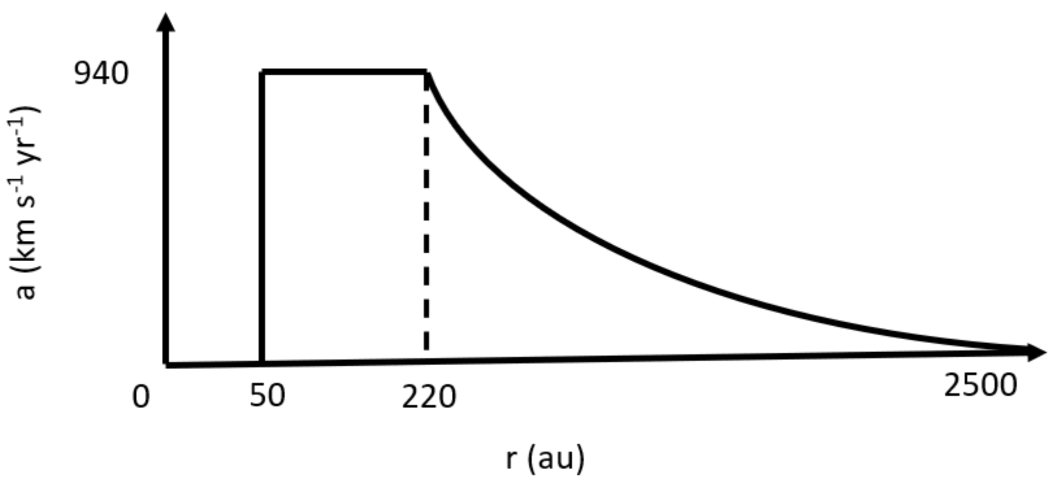
ii. Hence, sketch a(r) from $0 \le r \le 2500$ au.

Zero from $0 \le r < 50$ au

and flat section at ~940 km s^{-1} yr^{-1} from $50 \le r < 220$ au [1]

Following 1/r^2 shape from $220 \le r \le 2500$ au [1] [2]

[Lose 1 mark if the axes are not labelled, or a scale on either axis is missing. The horizontal axis does not need to be to scale, so long as key values indicated. Expect to see a (not to scale) sketch like the one below]



iii. Given shell 1 is measured to be moving at 2540 km s^{-1}, calculate the initial speed of the gas streaming away from the stars in the conical shock front, v_0, before it experiences the acceleration due to radiation pressure.

The work done by the radiative force on the dust is equal to its change in KE

$$\Delta KE = \frac{1}{2} m_{dust} v_{shell1}^2 - \frac{1}{2} m_{dust} v_0^2$$
$$= \int_{r_{inner}}^{r_{shell1}} F(r)\,\mathrm{d}r = m_{dust} \int_{r_{inner}}^{r_{shell1}} a(r)\,\mathrm{d}r$$
$$\therefore \frac{1}{2} v_0^2 = \frac{1}{2} v_{shell1}^2 - \int_{r_{inner}}^{r_{shell1}} a(r)\,\mathrm{d}r \qquad [1]$$

[This formula could also be arrived at by generalising the SUVAT equation $v^2 = u^2 + 2as$ into $v^2 = u^2 + 2\int a(s)\,\mathrm{d}s$]

In a. ii we were told the angular separation of shell 1 from the star, so

$$r_{shell1} = 1.64 \times \left(\frac{1.63}{3600} \times \frac{2\pi}{360}\right) = 1.30 \times 10^{-5}\ \mathrm{kpc} = 4.00 \times 10^{14}\ \mathrm{m} \qquad [1]$$

[This corresponds to 2670 au. If they use r_{shell1} = 2500 au in the subsequent integration lose only this mark (so get max 4 marks)]

Evaluating the integral with everything in SI units

$$\int_{r_{inner}}^{r_{shell1}} a(r)\,\mathrm{d}r = \int_{r_{inner}}^{r_{outer}} a_{max}\,\mathrm{d}r + \int_{r_{outer}}^{r_{shell1}} \frac{a_{max} r_{outer}^2}{r^2}\,\mathrm{d}r$$
$$= [a_{max} \times (r_{outer} - r_{inner})] + \left[-a_{max} r_{outer}^2 \left(\frac{1}{r_{shell1}^2} - \frac{1}{r_{outer}^2}\right)\right] \qquad [1]$$
$$= [2.99 \times 10^{-2} \times (220 - 50) \times 1.50 \times 10^{11}]$$
$$+[-2.99 \times 10^{-2} \times (220 \times 1.50 \times 10^{11})^2 \times \left(\frac{1}{(4.00\times10^{14})^2} - \frac{1}{(220\times1.50\times10^{11})^2}\right)$$
$$= 7.61 \times 10^{11} + 9.04 \times 10^{11}$$
$$= 1.67 \times 10^{12}\ \mathrm{m^2\ s^{-2}} \qquad [1]$$

[0.5 marks for either correctly evaluated integral (flat section or curved section)]

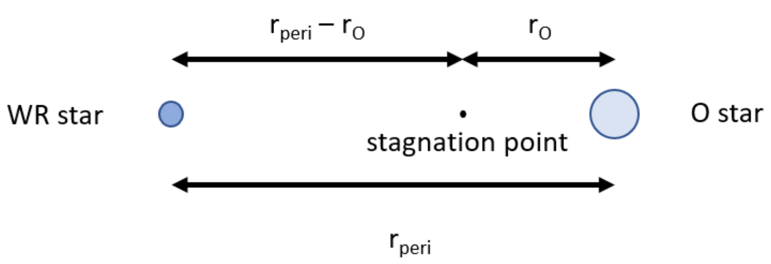
We can now evaluate v_0

$$\therefore v_0 = \sqrt{v_{shell1}^2 - 2\int_{r_{inner}}^{r_{shell1}} a(r)\,\mathrm{d}r} = \sqrt{(2540 \times 10^3)^2 - 2 \times 1.67 \times 10^{12}}$$
$$= 1.77 \times 10^6\ \mathrm{m\ s^{-1}} = \boxed{1770\ \mathrm{km\ s^{-1}}} \qquad [1] \qquad [5]$$

[Accept answer in m s^{-1} or km s^{-1} for final mark. Allow ecf from their a_{max}]

c. The measured semi-major axis of the WR140 system from radio interferometry is 8.82 milliarcseconds. Investigation of spectral lines shows that for the O star $v_{\infty,O}$ = 3100 km s^{-1} and for the WR star $v_{\infty,WR}$ = 2800 km s^{-1}, and measurements of X-ray absorption lead to suggested mass loss rates of $\dot{M}_O = 3.7 \times 10^{-7}\ M_\odot\ \mathrm{yr^{-1}}$ and $\dot{M}_{WR} = 1.7 \times 10^{-5}\ M_\odot\ \mathrm{yr^{-1}}$ respectively for each star. In the shock front, the pressure from each stellar wind is balanced. Considering the system at periapsis:

i. Show that the distance from the O star to the stagnation point is $r_O = \sqrt{\eta}/(1+\sqrt{\eta})\ r_{peri}$, where η is the ratio of the momenta of the two stellar winds, $\eta = p_O/p_{WR}$ and r_{peri} is the separation between the stars at periapsis.



The ram pressure is $p_{ram} = \frac{F}{A} = \frac{\dot{p}}{A}$ where $\dot{p}$ is the rate of change of momentum, so at the stagnation point (assuming spherically symmetric stellar winds)

$$\frac{\dot{p}_O}{4\pi r_O^2} = \frac{\dot{p}_{WR}}{4\pi\left(r_{peri}-r_O\right)^2} \quad [1]$$

$$\therefore \frac{\dot{p}_O}{\dot{p}_{WR}} = \frac{r_O^2}{\left(r_{peri}-r_O\right)^2}$$

But since $\eta = \frac{p_O}{p_{WR}} = \frac{\dot{p}_O}{\dot{p}_{WR}}$ then $\eta = \frac{r_O^2}{\left(r_{peri}-r_O\right)^2} \therefore \sqrt{\eta} = \frac{r_O}{r_{peri}-r_O}$ [1] [2]

This can then be rearranged to give the required formula

$$\sqrt{\eta}\left(r_{peri} - r_O\right) = r_O \therefore \sqrt{\eta}\,r_{peri} = \left(1+\sqrt{\eta}\right)r_O \therefore r_O = \frac{\sqrt{\eta}}{1+\sqrt{\eta}}r_{peri}$$

[The first mark is for setting up the problem algebraically **or** for a helpful diagram. The second mark is for correctly introducing η **and** having convincing algebra to the final answer (since it was given, no standalone mark for the final expression)]

ii. Hence, calculate r_O in au.

First, we can calculate η

$$\eta = \frac{p_O}{p_{WR}} = \frac{\dot{p}_O}{\dot{p}_{WR}} = \frac{\dot{M}_O v_{\infty,O}}{\dot{M}_{WR} v_{\infty,WR}} = \frac{3.7\times10^{-7}\times3100}{1.7\times10^{-5}\times2800} = 0.024 \quad [1]$$

We can also calculate the semi-major axis, and then use the given eccentricity to get the periapsis distance

$$a = 1.64 \times \left(\frac{8.82\times10^{-3}}{3600} \times \frac{2\pi}{360}\right) = 7.01\times10^{-8}\ \text{kpc} = 14.44\ \text{au} \quad [1]$$

$$r_{peri} = a(1-e) = 14.44 \times (1-0.8993) = 1.45\ \text{au} \quad [1]$$

Putting it all into the formula of the previous part

$$r_O = \frac{\sqrt{\eta}}{1+\sqrt{\eta}}r_{peri} = \frac{\sqrt{0.024}}{1+\sqrt{0.024}} \times 1.45 = \boxed{0.195\ \text{au}} \quad [1] \quad [4]$$

[Must be in au for the final mark, although a and r_{peri} might be given in metres (2.17×10^{12} m and 2.18×10^{11} m respectively)]

iii. Taking the WR star to be approximately 2 times more luminous than the O star, and the effective surface temperatures of the stars to be $T_{eff,O}$ = 35 kK and $T_{eff,WR}$ = 60 kK, calculate the radius of each star in solar units.

We can split up the luminosity calculated earlier in b. i.

$$L_{WR} = \frac{2}{3}L_{bin} = \frac{2}{3}\times1.43\times10^6 = 9.53\times10^5\ L_\odot = 3.67\times10^{32}\ \text{W} \quad [0.5]$$

$$L_O = \frac{1}{3}L_{bin} = \frac{1}{3}\times1.43\times10^6 = 4.77\times10^5\ L_\odot = 1.83\times10^{32}\ \text{W} \quad [0.5]$$

Using the Stephan-Boltzmann law

$$R_{WR} = \sqrt{\frac{L_{WR}}{4\pi\sigma T_{WR}^4}} = \sqrt{\frac{3.67\times10^{32}}{4\pi\times5.67\times10^8\times(60\times10^3)^4}} = 6.30\times10^9\ \text{m} = \boxed{9.06\ R_\odot} \quad [1] \quad [1.5]$$

$$R_O = \sqrt{\frac{L_O}{4\pi\sigma T_O^4}} = \sqrt{\frac{1.83\times10^{32}}{4\pi\times5.67\times10^8\times(35\times10^3)^4}} = 1.31\times10^{10}\ \text{m} = \boxed{18.82\ R_\odot} \quad [1] \quad [1.5]$$

[Score max 2 out of 3 if their final answer is not in $R_\odot$.

Give full credit if they assume $T_\odot \approx 5780$ K and so use $R = \sqrt{\frac{L/L_\odot}{(T/T_\odot)^4}}R_\odot$ for calculations]

[This matches up well with the scale of the stars in Figure 4 in the paper, showing these are both large, very hot stars.]

iv. Hence, determine for which star(s) the assumption of $v \approx v_\infty$ at the stagnation point is reasonable, and suggest (if any) the change in the position of the stagnation point taking in the real values of v at r_O.

Using the given formula:

$$v_{r_O} = v_{\infty,O}\left(1 - \frac{R_O}{r_O}\right) = \left(1 - \frac{1.31\times10^{10}}{0.195\times1.50\times10^{11}}\right)v_{\infty,O} = 0.55v_{\infty,O} \quad [1]$$

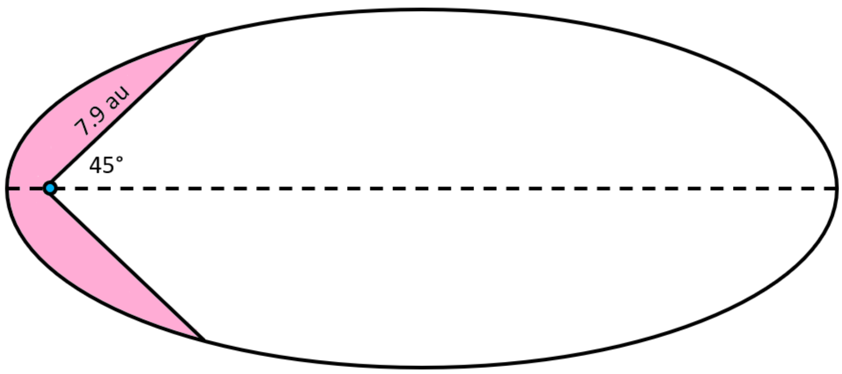
$$v_{r_{WR}} = v_{\infty,WR}\left(1 - \frac{R_{WR}}{r_{peri}-r_O}\right) = \left(1 - \frac{6.30\times10^{9}}{(1.45-0.195)\times1.50\times10^{11}}\right)v_{\infty,WR}$$
$$= 0.97v_{\infty,WR} \quad [1]$$

So $v \approx v_\infty$ is only a reasonable assumption for the WR star (at periapsis) [0.5]

This means the stagnation point will be even closer to the O star [0.5] [3]

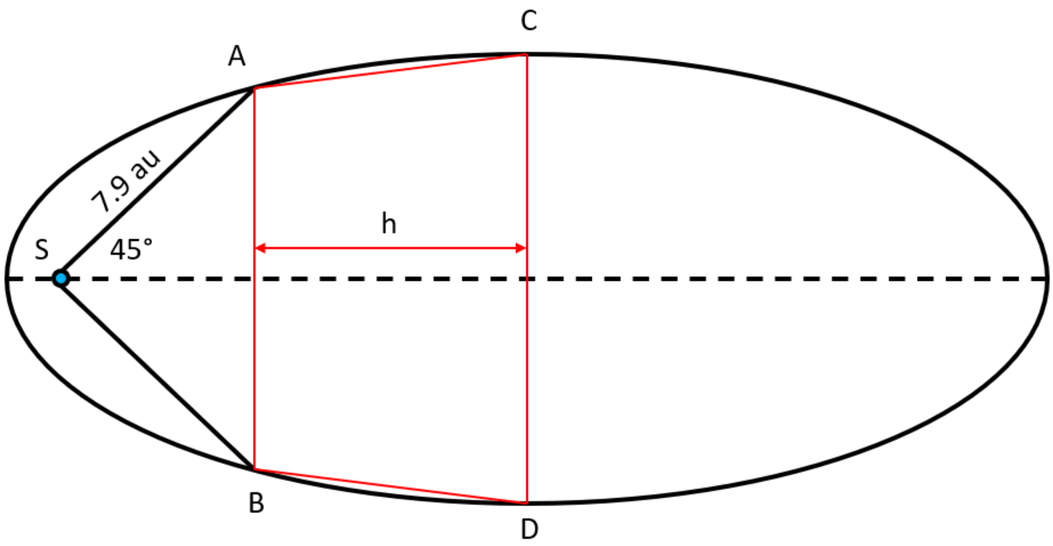
[Do not allow bald statement that it is a reasonable assumption for the WR star without evidence of calculation]

d. The conditions for dust production turn on when the stars get within ≈ 7.9 au of each other, corresponding to the solid red region in the right panel of Figure 4. A line from the WR star to the start of dust production region makes an angle of 45° with the semi-major axis. By using Kepler's second law, or otherwise, estimate the rough duration of dust production during each orbit. Give your answer in years to 2 decimal places, making any simplifying assumptions clear. [Hint: Kepler's second law states that the area swept out by one object orbiting another is equal in the same amount of time, such that dA/dt = constant = πab/T where T is the period]



From Kepler's 2nd law, $\frac{\text{area of shaded section}}{\text{total area of the ellipse}} = \frac{\text{time spent in shaded section}}{\text{period of the orbit}}$ [1]

We can use approximate geometry to calculate this area e.g. breaking the non-shaded section up into a triangle (ABS) and trapezium (ABDC) sat within a half-ellipse: [1]



Given the symmetry, angle $\angle ASB = 2 \times 45° = 90°$

$$\therefore \text{area of triangle } ABS = \frac{1}{2} \times 7.9^2 = 31.2 \text{ au}^2 \qquad [1]$$

We can also use Pythagoras' theorem to get the length AB

$$AB = \sqrt{7.9^2 + 7.9^2} = 11.2 \text{ au} \qquad [1]$$

The height of the trapezium

$$h = a - r_{peri} - 7.9 \cos 45° = 14.44 - 1.45 - 5.59 = 7.41 \text{ au} \qquad [1]$$

The length of CD is the minor axis (= 2b), which can be calculated from rearranging the formula given at the beginning of the paper

$$e = \sqrt{1 - \frac{b^2}{a^2}} \therefore b = a\sqrt{1 - e^2} = 14.44 \times \sqrt{1 - 0.8993^2} = 6.3 \text{ au} \qquad [1]$$

Hence, the area of the trapezium

$$\text{area } ABDC = \frac{1}{2}(AB + CD)h = \frac{1}{2}(11.2 + (2 \times 6.3)) \times 7.41 = 88.2 \text{ au}^2 \qquad [1]$$

Finally, the area of the half-ellipse (to the left of CD)

$$\text{area half-ellipse} = \frac{1}{2}\pi ab = \frac{1}{2}\pi \times 14.44 \times 6.3 = 143.4 \text{ au}^2 \qquad [1]$$

The area of the shaded section is therefore

$$\text{shaded area} = \text{area of half-ellipse} - (\text{area ABS} + \text{area ABDC})$$
$$= 143.4 - (31.2 + 88.2) = 24.0 \text{ au}^2 \qquad [1]$$

Finally, applying Kepler's 2nd law

$$t = \frac{\text{shaded area}}{\text{total area}} \times T = \frac{24.0}{2 \times 143.4} \times 2895 = 0.084 \times 2895 = 242 \text{ days} = \boxed{0.66 \text{ yrs}} [1] \qquad [10]$$

[Must be in years and to 2 d.p. for the final mark. A wide variety of approaches are possible, so award marks based upon sensible choices of shapes to evaluate the area of. The first mark is for realising how to apply Kepler's 2nd law in this context, the second mark is for a clear approach to break the ellipse up into shapes (ideally with a diagram – and not necessarily the triangle and trapezium used here), and the final mark is for the final answer – all other marks are available for sensible areas and lengths relevant to their method. Accept any method that gives a value of 0.60-0.70 years so long as the justification is clear]

[The more exact way to work this out is using our knowledge of anomalies, although some approximations were made to simplify the geometric case in terms of the given length and angle. Assuming the angle to be correct, leads to a true anomaly $\nu = 180° - 45° = 135°$. Given this and the quoted eccentricity and semi-major axis, the real length AS is 7.6 au, and utilising Kepler's equation to get the time for $-135° \leq \nu \leq 135°$ gives 0.633 years. Alternatively, assuming the length AS = 7.9 au to be correct, leads to a real true anomaly of 136.3°, and when looking at the time to travel from $-136.3° \leq \nu \leq 136.3°$ gives 0.671 years. Students using this approach (rather than areas of shapes) get full credit. However the problem is approached, it is clear that dust is only produced on a relatively short time frame, which is why each shell has such contrast.]

Q3 – Planetary Migration in Protoplanetary Discs [25 marks]

a. Consider a planet of mass M_p in a circular orbit of radius r_p about a star of mass M_*, with $M_p/M_* = q << 1$. Show that it has angular velocity and angular momentum about the central star given by $\Omega_p = \sqrt{(GM_*/r_p^3)}$, and $L = M_p\sqrt{(GM_*r_p)}$

Balancing the centripetal force with the gravitational force [accept balancing accelerations]

$$M_p r_p \Omega_p^2 = \frac{GM_* M_p}{r_p^2}$$ [1] [1]

Cancelling M_p and rearranging gives $\Omega_p = \sqrt{\frac{GM_*}{r_p^3}}$ as required

[This was a 'show that', so the mark is for the algebraic route to the answer, not the final answer]

Using the equations for the hint at the end of the question

$$L = I\Omega_p = M_p r_p^2 \Omega_p$$ [1] [1]

Substituting the expression for Ω_p gives $L = M_p\sqrt{GM_* r_p}$ as required

[This was a 'show that', so the mark is for the algebraic route to the answer, not the final answer]

b. Consider now a rotationally symmetric disc with surface density profile $\Sigma = \Sigma_0(r/r_0)^{-3/2}$, and outer radius $r_{out} = 9r_0$. Find the mass of the disc M_{disc} in terms of Σ_0 and r_0 assuming its inner radius $r_{in} << r_0$.

We will need to integrate the surface density (mass per unit area) over the total area of the disc

$$M_{disc} = \int_{r_{in}}^{r_{out}} \Sigma \, dA$$ [1]

Given $r_{out} = 9r_0$ and $r_{in} << r_0$ our lower integral limit is ≈ 0, and $dA = 2\pi r \, dr$ so

$$M_{disc} = \int_0^{9r_0} \Sigma_0 \left(\frac{r}{r_0}\right)^{-3/2} 2\pi r \, dr$$ [1]

$$= \left[\Sigma_0 r_0^{3/2} 4\pi r^{1/2}\right]_0^{9r_0}$$ [1]

$$= 12\pi\Sigma_0 r_0^2$$ [1] [4]

[If the first mark is not already given, it can be scored for either a correct approximation of 0 for the lower integral limit, or for converting from dA to dr. Students that do $M_{disc} = \int_0^{9r_0} \Sigma \, dr = -\frac{2}{3}\Sigma_0 r_0$ get a maximum of 1 mark (for the lower integral limit), since the negative total mass is non-physical]

c. Show that for a disc of uniform entropy, $\Gamma = \Gamma_L + \Gamma_C = (5.76 - (5.11 + 2.33\gamma)\alpha)\Gamma_0$.

The final expression does not include δ, so we need to find it as a function of α and γ so we can combine the given expressions. Given $\Sigma \propto r^{-\alpha}$ and $P \propto r^{-\delta}$,

$$\frac{P}{\Sigma T} = \text{constant} \therefore T \propto \frac{P}{\Sigma} \propto \frac{r^{-\delta}}{r^{-\alpha}} \Rightarrow T \propto r^{-\delta+\alpha} \quad (1)$$ [1]

We are told it is a disc of uniform entropy (so ds = 0)

$$ds = \text{constant} \times \left(\frac{1}{\gamma-1}\frac{dT}{T} - \frac{d\Sigma}{\Sigma}\right) = 0 \therefore \frac{dT}{T} = (\gamma - 1)\frac{d\Sigma}{\Sigma}$$ [1]

Using the hint given

$$\Rightarrow T \propto \Sigma^{\gamma-1}$$ [1]

But given $\Sigma \propto r^{-\alpha}$ from earlier

$$\Rightarrow T \propto r^{-\alpha\gamma+\alpha} \quad (2)$$ [1]

Comparing (1) and (2)

$$\Rightarrow \delta = \alpha\gamma$$ [1] [5]

Substituting this into the given expressions for Γ_L and Γ_C gives the required expression

$$\Gamma = \Gamma_L + \Gamma_C = (-3.20 + 0.86\alpha - 2.33\alpha\gamma + 5.97 \times 1.5 - 5.97\alpha)\Gamma_0$$

$$= (5.755 - (5.11 + 2.33\gamma)\alpha)\Gamma_0$$ ['Show that' so mark for route not answer]

d. Assume the disc model from part b. with h = const. Find the time taken for the orbital radius of the planet to halve from initial radius r_0. Take γ = 1.4 and give your answer in terms of the migration timescale, $t_m = 1/h\,(h^3/q)\, M_*/M_{disc}\, \tau_0$, where τ_0 is the orbital period at radius r_0.

Using our earlier expression for L and assuming only the orbital radius changes with time

$$\Gamma = \frac{dL}{dt} = \frac{d}{dt}\left(M_p\sqrt{GM_* r_p}\right) = \frac{1}{2}M_p\sqrt{GM_*}r_p^{-1/2}\frac{dr_p}{dt} \qquad [1]$$

In the model of part b., $\Sigma \propto r^{-3/2}$ so by comparison with part c. $\alpha = 3/2$, and we are told $\gamma = 1.4$
Using the expression from part c. for Γ and inputting Γ_0 with the earlier expression for Ω_p

$$\Gamma = (5.76 - (5.11 + 2.33\gamma)\alpha)\Gamma_0$$

$$= (5.76 - (5.11 + 2.33 \times 1.4) \times 1.5) \times \left(\frac{q}{h}\right)^2 \Sigma_p r_p^4 \frac{GM_*}{r_p^3} \qquad [1]$$

$$= -6.80\left(\frac{q}{h}\right)^2 \Sigma_p GM_* r_p \qquad [1]$$

Equating these two expressions for Γ and rearranging for dr_p/dt

$$\frac{1}{2}M_p\sqrt{GM_*}r_p^{-1/2}\frac{dr_p}{dt} = -6.80\left(\frac{q}{h}\right)^2 \Sigma_p GM_* r_p$$

$$\therefore \frac{dr_p}{dt} = -13.60\left(\frac{q}{h}\right)^2 \Sigma_p \frac{\sqrt{GM_*}}{M_P} r_p^{3/2} \qquad [1]$$

However, we recognise that $\Sigma_p = \Sigma_0\left(\frac{r_p}{r_0}\right)^{-3/2}$

$$\therefore \frac{dr_p}{dt} = -13.60\left(\frac{q}{h}\right)^2 \Sigma_0 \frac{\sqrt{GM_*}}{M_P} r_0^{3/2} \qquad [1]$$

All of the parameters on the right hand side are constant $\therefore \frac{dr_p}{dt} =$ constant, so if in time t the radius of the planet has halved, $\frac{dr_p}{dt} = -\frac{\frac{1}{2}r_0}{t}$ [negative since r_p is decreasing] [1]

Substituting it in and rearranging for t

$$-\frac{\frac{1}{2}r_0}{t} = -13.60\left(\frac{q}{h}\right)^2 \Sigma_0 \frac{\sqrt{GM_*}}{M_P} r_0^{3/2}$$

$$\therefore t = \frac{1}{27.19}\left(\frac{h}{q}\right)^2 \frac{1}{\Sigma_0}\frac{M_p}{\sqrt{GM_*}}\frac{1}{\sqrt{r_0}} \qquad [1]$$

From Kepler's 3rd law, we can derive an expression for the initial orbital period, τ_0

$$\tau_0 = \frac{2\pi}{\sqrt{GM_*}} r_0^{3/2}$$

$$\therefore t = \frac{1}{27.19}\left(\frac{h}{q}\right)^2 \frac{M_p}{2}\frac{\tau_0}{\pi\Sigma_0 r_0^2} \qquad [1]$$

As found in part b., $M_{disc} = 12\pi\Sigma_0 r_0^2$ and remembering that $q = M_p/M_*$

$$\therefore t = \frac{1}{27.19}\left(\frac{h}{q}\right)^2 \frac{M_p}{2}\frac{12\tau_0}{M_{disc}} = \frac{6}{27.19}\frac{1}{h}\frac{h^3}{q}\frac{M_*}{M_{disc}}\tau_0 = \boxed{0.221 t_m} \qquad [1] \qquad [9]$$

[Must be in terms of the migration timescale, t_m, for the final mark. The numerical factor may be left as 1/4.53 or equivalent rather than evaluated]

e. If the disc has mass $M_{disc} = 0.01M_*$ and aspect ratio h = 0.05, and if $q = 5 \times 10^{-6}$ and τ_0 = 10 years, find the elapsed time in years for the migration described in part d. to occur. Is this a feasible mechanism for planetary migration given the lifetime of the disc is roughly 10 Myr?

Substituting the numbers into the equation we just derived

$$t = 0.221 \times \frac{1}{0.05} \times \frac{0.05^3}{5\times10^{-6}} \times \frac{1}{0.01} \times 10 \text{ yrs} = \boxed{110\,000 \text{ yrs}} \qquad [1] \qquad [1]$$

Since t < 10 Myr then $\boxed{\text{yes}}$, it is a feasible mechanism [1] [1]

[Award 0.5 marks for calculating t_m (= 0.5 Myr). Second mark can only be gained with a sensible comparison between their t (allowing ecf) and 10 Myr, not for a bald statement. Allow use of the more precise formula for Γ from c. leading to 110 246 years rather than 110 327 years]

f. Whilst the planet excites a spiral density wave over the whole disc, in fact the dominant contribution to the torque from the spiral is from only a small neighbourhood around the planet, so that α may be replaced by $-r_p/\Sigma_p \, d\Sigma/dr|_p$. Near the inner radius of the disc, the surface density departs from the above power-law behaviour and smoothly decreases to 0. Does the migration stop before the planet reaches the disc's inner radius?

The migration will stop when Γ = 0, which we can use to define a critical value of α

$$\Gamma = (5.76 - (5.11 + 2.33\gamma)\alpha_{crit})\Gamma_0 = 0$$

$$\therefore \alpha_{crit} = \frac{5.76}{5.11+2.33\gamma} \qquad [1]$$

Since $\gamma \geq 1$ this means $0 < \alpha_{crit} < 0.774$

Far from the inner edge (as used previously in the question), $\alpha = 3/2 = 1.5$

We are told that Σ decreases to zero at the inner edge, so the variation of Σ as a function of increasing r must be that it starts from zero at r_{inner}, increases smoothly to some maximum, and then decreases again following $\Sigma \propto r^{-3/2}$.

Since $\alpha = -\frac{r_p}{\Sigma_p}\frac{d\Sigma}{dr}\Big|_p$ then following the behaviour described above there must be a region near r_{inner} where $\frac{d\Sigma}{dr}\Big|_p$ is positive (since in that region Σ is increasing with increasing r) and so (since r_p and Σ_p are also positive) α is negative [1]

[This mark is for any reasoning that to match the described behaviour then there must be a change in the sign of α as you approach r_{inner}]

This means far from the inner edge $\alpha = 1.5 >$ all possible α_{crit}

and close to the inner edge $\alpha < 0 <$ all possible α_{crit}

so there must be a place in the disc between r_{outer} and r_{inner} where $\alpha = \alpha_{crit}$

hence the migration does halt before the inner edge of the disc [1] [3]

[No mark for a bald statement without some attempt at mathematical justification]

[This is the reason planets are able to form **and** survive to be detected in mature star systems, otherwise planets would only be a feature of young systems (before spiralling into their star) and life on Earth would not have had enough time to evolve]