BAAO Astro Challenge Astrophysics Self-Study Guide

Extra things beyond A Level that you should know to help with answering questions in the Astro Challenge

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Elliptical Orbits

At A Level you will only ever encounter circular orbits. Whilst they are a good approximation for the orbits of many objects, in reality the orbits are elliptical in shape.

The BAAO standard diagram to explain different features of an elliptical orbit is shown below:

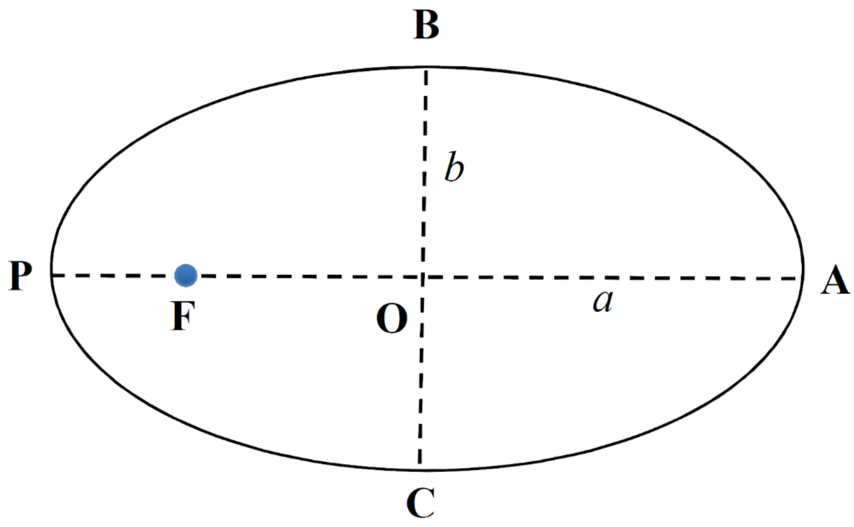


Figure 1 *The main aspects of an elliptical orbit*

Features of an elliptical orbit

Point F is one of the foci of the ellipse (the other is symmetrically between O and A). This is where the centre of mass of the system is, which typically for a planet-star system is well approximated as the centre of the star.

> Kepler's 1st Law is that the orbit of a planet is an ellipse with a star at one of the two foci; this could be extended and generalised to say that in any gravitationally bound system, objects will follow an elliptical orbit with the centre of mass at one of the foci

The **major** axis is the longest axis of the ellipse (i.e. PA), and half that distance is the **semi-major** axis, a, and so $a = OP = OA$

The **minor** axis is the shortest axis of the ellipse (i.e. BC), and half that distance is the **semi-minor** axis, b, and so $b = OB = OC$

The relationship between the semi-major and semi-minor axes is given by the eccentricity of the ellipse, e, defined as

$$e = \sqrt{1 - \frac{b^2}{a^2}}$$

As a becomes much larger than b, the eccentricity approaches 1, whilst when $a = b$ then $e = 0$; in other words, a circle has an eccentricity of zero.

The shortest distance between an orbiting object and the focus is called the periapsis (in general) and in Figure 1 this is the distance PF. It is related to the semi-major axis and eccentricity as

$$r_p = a(1 - e)$$

The greatest distance between an orbiting object and the focus is called the apoapsis (in general) and in Figure 1 this is the distance AF. It is related to the semi-major axis and eccentricity as

$$r_a = a(1 + e)$$

The prefixes peri- and apo- can be applied to specific objects to change the ending of the word – common ones include perigee and apogee (for the Earth) and perihelion and aphelion (for the Sun); general rule of thumb is look for the prefix to work out what the question is talking about.

In the left-hand side of the figure below you can see the effect of changing the eccentricity but keeping the semi-major axis the same – all these orbits would therefore have the **same** period. By contrast, in the right-hand side of the figure you can see the four main categories of orbit, where each one has a different eccentricity and **also** a different semi-major axis (chosen to keep the periapsis the same). This is more like the orbital changes you find in most questions caused by a probe firing its rockets to either speed it up or slow it down at a given point in space and so move from one shape of orbit to another one. Hyperbolas will only be examined in BAAO Round 2.

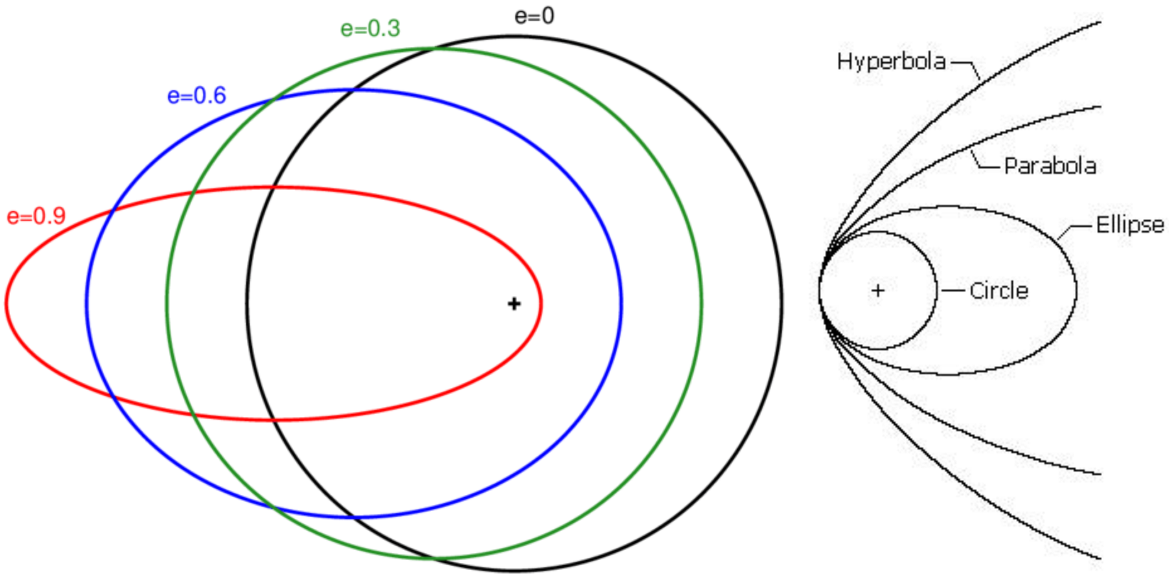


***Figure 2** Left: Varying eccentricity but keeping the semi-major axis the same. Right: Varying eccentricity but keeping the periapsis the same.*

The area of an ellipse is related to the semi-major and semi-minor axes as

$$A = \pi ab$$

showing that for a circle (where $r = a = b$) the familiar equation for the area of a circle is recovered. Note that there is no closed form for the circumference of an ellipse (unlike for a circle).

Total energy in an orbit

All bound orbits (i.e. circular or elliptical) have a negative total energy, whilst (by definition) objects on a parabolic orbit are travelling at escape velocity and so have a total energy of zero.

For a circular orbit:

$$E_{tot} = GPE + KE = -\frac{GMm}{r} + \frac{1}{2}mv^2 \quad \text{but} \quad v = \sqrt{\frac{GM}{r}} \quad \therefore E_{tot} = -\frac{GMm}{r} + \frac{1}{2}m\left(\frac{GM}{r}\right) = \boxed{-\frac{1}{2}\frac{GMm}{r}}$$

A circular orbit can become an elliptical one if the speed of the object at a given point in the orbit is changed, for example via use of rockets. If it is sped up then the point where it fired its rockets becomes the new periapsis, whilst if it is slowed down then that point become the new apoapsis. The new total energy of the orbit is the **same** as if it was a **circular** orbit with a radius equal to the semi-major axis, namely

$$E_{tot} = \boxed{-\frac{1}{2}\frac{GMm}{a}}$$

Therefore, as the orbit becomes more and more eccentric, in the limit $a \to \infty$ as $e \to 1$ then $E_{tot} \to 0$, as expected for a parabolic orbit.

This allows us to summarise the energy of different orbits as:

- Circular: $E_{tot} = -\frac{1}{2}\frac{GMm}{r}$
- Elliptical: $E_{tot} = -\frac{1}{2}\frac{GMm}{a}$
- Parabolic: $E_{tot} = 0$

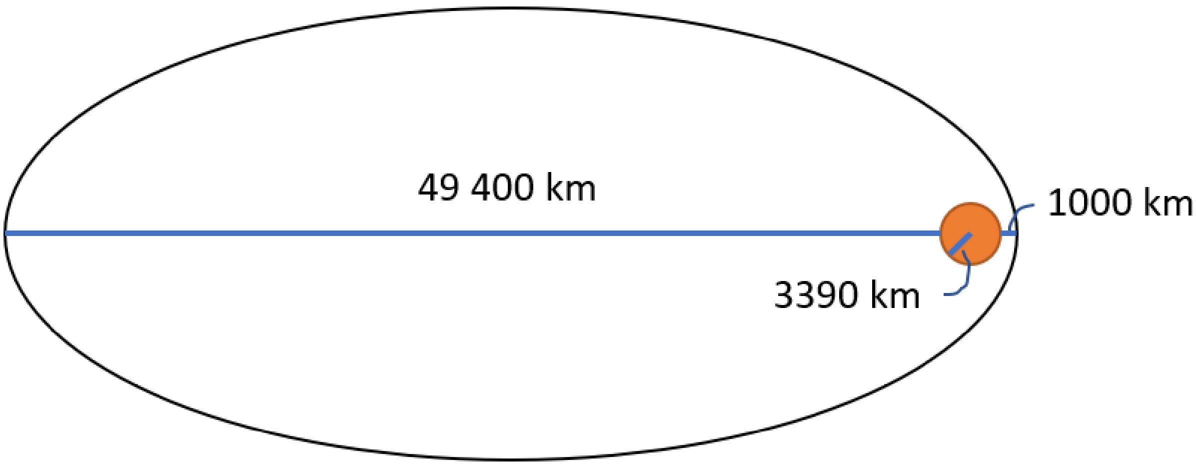
Worked Examples

1. A comet is orbiting the Sun in an orbit with a semi-major axis of 5.0 au and an eccentricity, e = 0.80. What is the area of the orbit?

$$e = \sqrt{1 - \frac{b^2}{a^2}} \therefore b = a\sqrt{1 - e^2} = 5.0\sqrt{1 - 0.80^2} = 3.0 \text{ au}$$

$$A = \pi ab = \pi \times 5.0 \times 3.0 = 15\pi = 47 \text{ au}^2$$

2. The Hope probe from the UAE orbits Mars with an orbit that ranges from 49 400 km away from the surface to as close as only 1000 km above the planet. Given the mass of Mars is 6.39 × 10^{23} kg and its radius is 3390 km, what is the period of this orbit?



$$r_p = 3390 + 1000 = 4390 \text{ km} \quad \text{and} \quad r_a = 49400 + 3390 = 52790 \text{ km}$$

$$a = \frac{1}{2}(r_p + r_a) = \frac{1}{2}(4390 + 52790) = 28590 \text{ km}$$

$$T = \sqrt{\frac{4\pi^2}{GM}a^3} = \sqrt{\frac{4\pi^2}{6.67 \times 10^{-11} \times 6.39 \times 10^{23}} \times (28590 \times 10^3)^3} = 1.47 \times 10^5 \text{ s } (= 40.9 \text{ hours})$$

3. How much energy would the Moon need to gain to leave Earth's orbit?
[The mass of the Earth is 5.97 × 10^{24} kg and mass of the Moon is 7.35 × 10^{22} kg. The semi-major axis of the Moon's orbit is 384 400 km]

$$E_{tot} = -\frac{1}{2}\frac{GMm}{a} = -\frac{1}{2}\frac{6.67 \times 10^{-11} \times 5.97 \times 10^{24} \times 7.35 \times 10^{22}}{384400 \times 10^3} = -3.81 \times 10^{28} \text{ J}$$

If on an escape orbit then E_{tot} = 0, and so the energy the moon needs to gain is 3.81 × 10^{28} J

4. [Round 1 level] A comet has a perihelion that is half its semi-**minor** axis. What is its eccentricity?

$$r_p = a(1-e) = \frac{1}{2}b \quad \text{and} \quad b = a\sqrt{1-e^2}$$
$$\therefore 1 - e = \frac{1}{2}\sqrt{1-e^2}$$
$$\therefore (1-e)^2 = \frac{1}{4}(1-e^2)$$
$$\therefore 4 - 8e - 4e^2 = 1 - e^2$$
$$\therefore 3e^2 + 8e - 3 = 0$$
$$\therefore (3e-1)(e+3) = 0$$
$$\therefore e = \frac{1}{3} \quad \text{or} \quad e = -3$$
$$\text{Since only positive solutions are allowed,} \qquad e = \frac{1}{3}$$

5. [Round 1 level] A satellite of mass 200 kg orbiting Earth 400 km above the surface in a circular orbit fires its engines. [The radius of the Earth is 6370 km]
   a. If the rockets accelerate it so it gains 2.0 GJ, what is the apogee distance above the surface of the Earth in the new orbit?
   b. If the rockets had instead decelerated it, how much energy would it need to lose in order to crash into Earth?

Part a)
Calculating the initial energy of the orbit

$$E_{initial} = -\frac{1}{2}\frac{GMm}{r} = -\frac{1}{2}\frac{6.67 \times 10^{-11} \times 5.97 \times 10^{24} \times 200}{(6370+400) \times 10^3} = -5.88\ \text{GJ}$$

After the burn by the rocket, $E_{final} = -3.88\ \text{GJ}$

The new semi-major axis will therefore be

$$a = -\frac{1}{2}\frac{GMm}{E_{final}} = -\frac{1}{2}\frac{6.67 \times 10^{-11} \times 5.97 \times 10^{24} \times 200}{-3.88 \times 10^9} = 10258\ \text{km}$$

Since $a = \frac{1}{2}(r_p + r_a)$ and the initial radius of the circular orbit is now the perigee distance

$$r_a = 2a - r_p = (2 \times 10258) - (6370 + 400) = 13746\ \text{km}$$

So, distance above the surface of the Earth $= 13746 - 6370 = 7376 \approx 7380\ \text{km}$

Part b)
The limiting orbit for it to crash is if its new perigee distance is equal to the radius of the Earth (given the original orbit distance is now the new apogee)

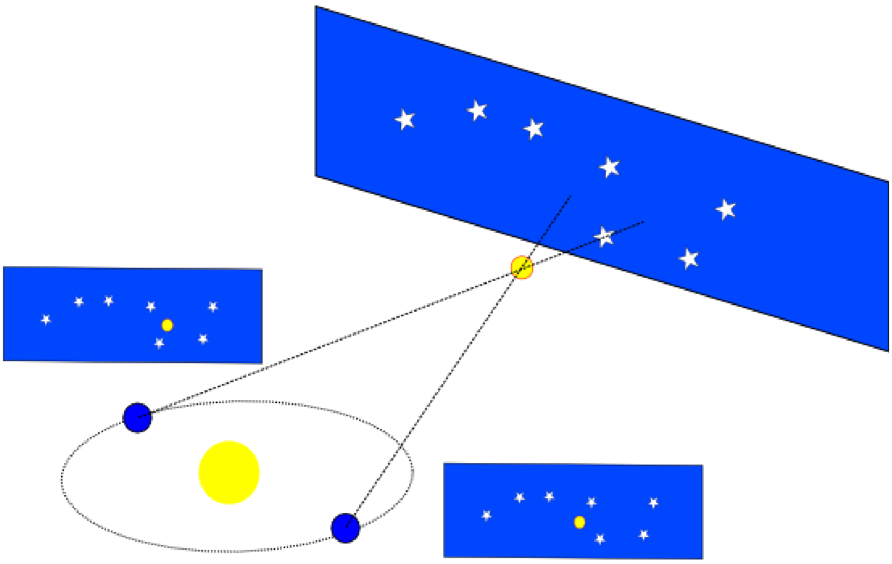
$$\therefore a = \frac{1}{2}(r_p + r_a) = \frac{1}{2}\big(6370 + (6370+400)\big) = 6370 + 200 = 6570\ \text{km}$$
$$\therefore E_{tot} = -\frac{1}{2}\frac{GMm}{r} = -\frac{1}{2}\frac{6.67 \times 10^{-11} \times 5.97 \times 10^{24} \times 200}{6570 \times 10^3} = -6.06\ \text{GJ}$$

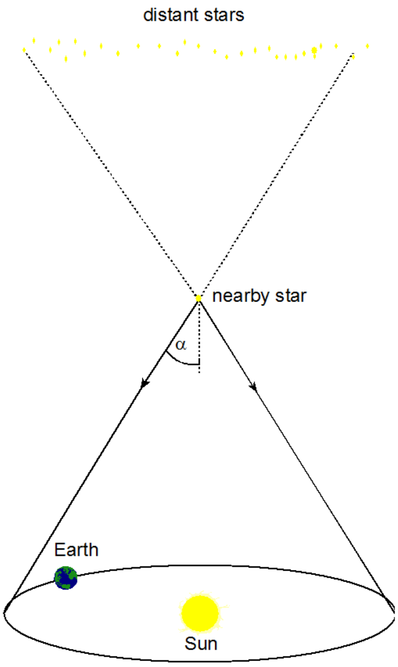
So, the energy that needs to be lost is $(6.06 - 5.88) = 0.18\ \text{GJ}$

Parallax

For astronomical scales a metre is not a very helpful measurement of distance, so there are three new units that you will encounter in BAAO competitions:

1. ASTRONOMICAL UNIT (au)
   1 au = the average distance from the Earth to the Sun
   = 1.50×10^{11} m
2. LIGHT YEAR (ly)
   1 ly = distance travelled by light in 1 year
   = $(3.00 \times 10^{8}) \times (365 \times 24 \times 60 \times 60)$
   = 9.46×10^{15} m
3. PARSEC (pc)
   1 pc = 3.26 ly = 3.09×10^{16} m
   (we will define this unit properly very shortly)





For reasonably nearby stars we can use the method of parallax. This involves measuring the change in position of a star relative to the background stars

The reason for this apparent change in position is that the Earth has moved in its orbit around the Sun

[It is similar to the way your finger appears to move if you stick it out in front of you and look with one eye at a time – this is in fact how your brain calculates distances to objects and makes your world 3D!]

Typically, the measurements are taken 6 months apart to get the biggest change in position. It is the angular movement that is measured.

Small angles

Since all stars are distant this effect is tiny, and so the angles involved are very small

Circle = 360 degrees (°)

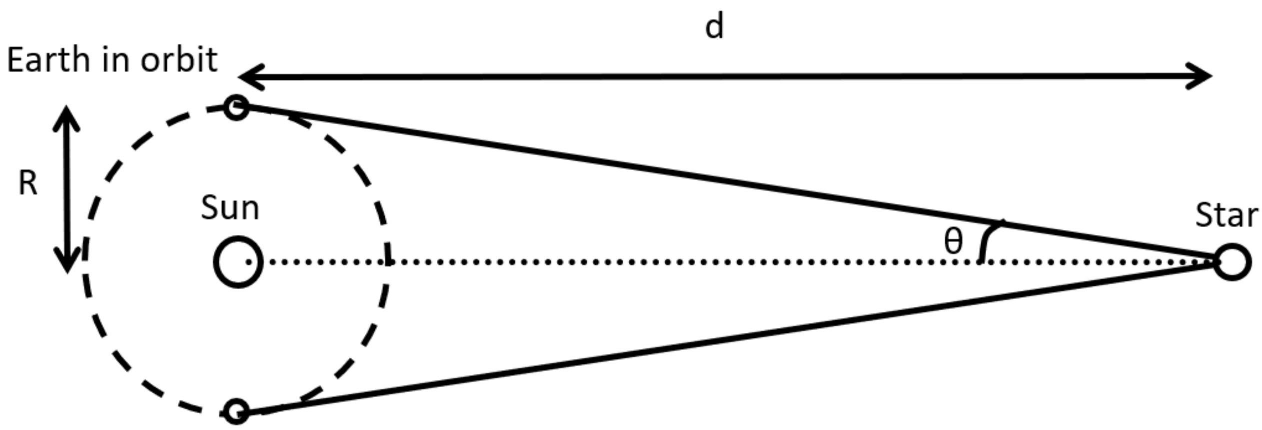
1° = 60 arcminutes (‘)

1’ = 60 arcseconds (“)

It is quite common to see SI prefixes used with arcseconds, most notably milliarcseconds (mas = 10^{-3} arcseconds) and microarcseconds (μas = 10^{-6} arseconds)

Definition of a parsec

Consider the geometry of the situation:

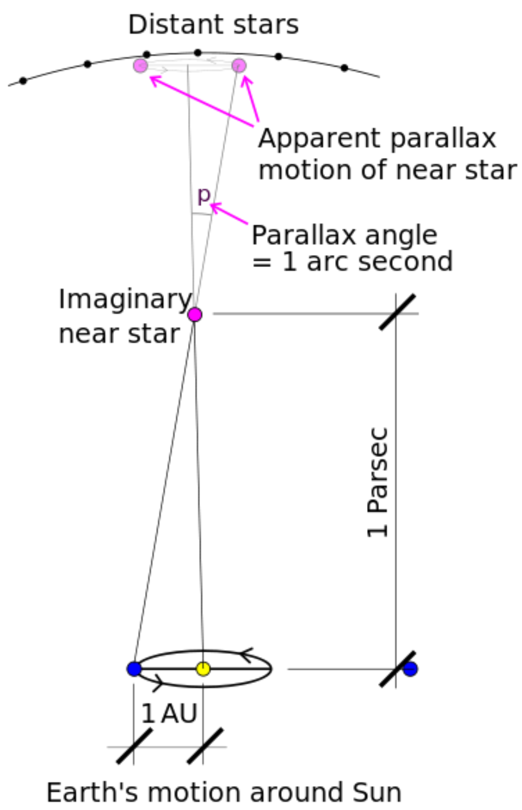


From trigonometry $\tan\theta = \frac{R}{d}$, and since θ is very small we can use the small angle approximation $\tan\theta \approx \theta$ and so $\theta = \frac{R}{d}$ where θ is in radians.

The name 'parsec' is an abbreviation of 'parallel angle of one arcsecond' and is defined as the distance d for which $\theta = 1''$ when $R = 1$ au.

Due to the definition of the parsec this leads to very convenient formula for distance in parsecs if you know the (observed) parallax angle in arcseconds (and vice versa):

$$d = \frac{1}{p}$$



Worked Examples

1. From the trigonometric definition of a parsec, calculate a parsec in metres.

$$d = \frac{R}{\theta} = \frac{1 \text{ au in m}}{1\text{" in radians}} = \frac{1.50 \times 10^{11}}{\left(\frac{2\pi}{360} \times \frac{1}{60} \times \frac{1}{60}\right)}$$
$$= 3.09 \times 10^{16} \text{ m}$$

2. The star Achernar is 1.32 × 10^{18} m away. Calculate its parallax angle in milliarcseconds.

$$d = \frac{1.32 \times 10^{18}}{3.09 \times 10^{16}} = 42.7 \text{ pc}$$
$$p = \frac{1}{d} = \frac{1}{42.7} = 0.0234\text{"} = 23.4 \text{ mas}$$

Magnitudes

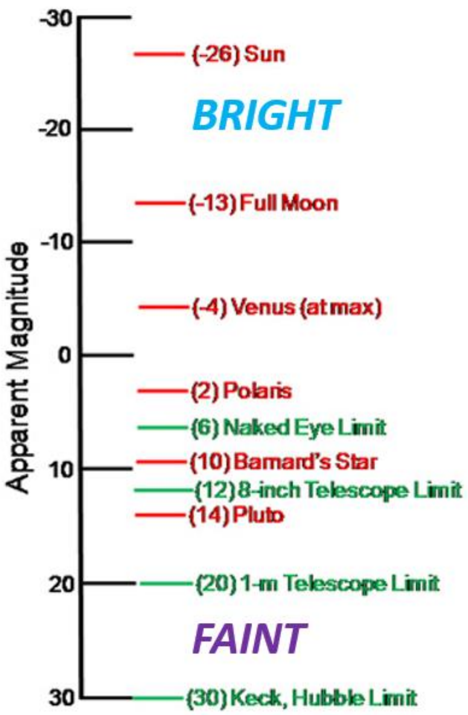
Apparent Magnitude

The ancient Greeks invented a scale over 2000 years ago to classify the brightness of stars in the sky, numbering them from 1 (brightest) to 6 (faintest).

This led to the magnitude scale used by astronomers today. Due to the way the eye responds in low light, the scale is logarithmic, and a magnitude 1 star is **defined** to be 100 times brighter than a magnitude 6 star, so every step on the scale (or which there are 5) corresponds to a change in brightness of $100^{1/5} = 2.512$.

On this scale large numbers correspond to faint objects and small (or negative) numbers correspond to bright ones, so an object with an apparent magnitude of +14 is very faint, +2 is bright, and -4 is exceptionally bright.

BAAO questions will assume a limiting magnitude for the naked eye in a (very) dark sky to be +6, unless told otherwise.



Since an increase of 5 magnitudes corresponds (by definition) to an increase in brightness of a factor of 100, we can generalise this to give the formula

$$\frac{b_1}{b_0} = 100^{0.2(m_0 - m_1)} = 10^{-0.4(m_1 - m_0)}$$

[We can easily see that if we input $m_1 = 1$ and $m_0 = 6$ into either formula then $b_1 = 100 b_0$ as expected]

BAAO questions will use a script-style m to indicate apparent magnitude – this is to distinguish it from m, which could be used to mean e.g. mass.

This formula can of course be inverted to give

$$m_1 - m_0 = -2.5 \log_{10}\left(\frac{b_1}{b_0}\right)$$

[Note that the 2.5 at the front is from the fraction 5/2 and so is NOT to be confused with $100^{0.2}$ = 2.512]

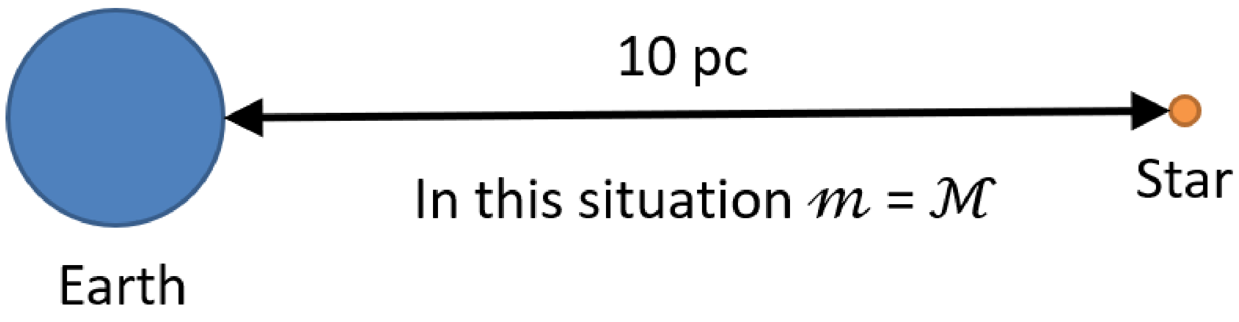
BAAO mark schemes will generally be written in terms of base 10 logarithms, and you will not be expected to use the base 100 definition, although these formulae are all mathematically equivalent so if you feel more comfortable thinking in steps of size 2.512 that is fine, and you would still be eligible for all the marks.

Absolute Magnitude

The apparent magnitude is a measure of how bright a star appears when observed in the sky, but this will clearly change with distance since if you get closer to an object its apparent magnitude will be brighter, and so m is not a measure of the star's intrinsic brightness.

To do that we will define an absolute magnitude:

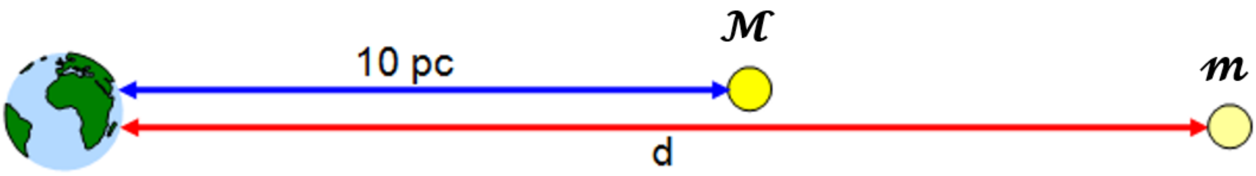
> The absolute magnitude of a star, $\mathcal{M}$, is the apparent magnitude the star would have from a distance of 10 pc



Again, here we can see that BAAO questions will use a script-style $\mathcal{M}$ to indicate absolute magnitude – this is to distinguish it from M, which is often used to mean mass.

The apparent magnitude of a star, m, its absolute magnitude, $\mathcal{M}$, and the distance to the star, d (in parsecs), are related by the formula:

$$m - \mathcal{M} = 5\log_{10}\left(\frac{d}{10}\right) = 5\log_{10} d - 5$$



This also means that two stars that are the same distance away from an observer (e.g. in a binary pair where the separation between them is far less than the distance to the binary system) then

$$m_A - m_B = \mathcal{M}_A - \mathcal{M}_B$$

Generalising this further by considering the second formula of the previous section, we can also link absolute magnitudes to luminosities, since $b = \frac{L}{4\pi d^2}$

$$\mathcal{M}_1 - \mathcal{M}_0 = -2.5\log_{10}\left(\frac{b_1}{b_0}\right) = -2.5\log_{10}\left(\frac{\frac{L_1}{4\pi d^2}}{\frac{L_0}{4\pi d^2}}\right) = -2.5\log_{10}\left(\frac{L_1}{L_0}\right)$$

This can also be very helpfully done by using the Sun as a defined reference point ($\mathcal{M}_\odot = 4.74$) to turn absolute magnitudes into luminosities (in units of solar luminosity, $L_\odot = 3.83 \times 10^{26}$ W), which could in turn be converted into SI units or used to calculate brightness etc.

Worked Examples

1. Star A has an apparent magnitude of -0.37 and star B has an apparent magnitude of 4.92. How many times brighter is star A than star B?

$$\frac{b_A}{b_B} = 10^{-0.4(m_A - m_B)} = 10^{-0.4(-0.37-4.92)} = 131$$

2. What is the luminosity (in solar luminosities) of a star with an absolute magnitude of 0?

We can calculate this by knowing that the Sun has a defined absolute magnitude of 4.74 and so everything can be determined relative to that

$$0 - \mathcal{M}_\odot = -2.5\log_{10}\left(\frac{L}{L_\odot}\right) \therefore L = 10^{\frac{4.74}{2.5}} L_\odot = 78.7 L_\odot$$

3. What is the brightness of a star with an apparent magnitude of 0?

Again, keeping things relative to the Sun, and knowing that this would correspond to the brightness of a star with an absolute magnitude of 0 at 10 pc

$$b_0 = \frac{L_0}{4\pi d^2} = \frac{78.7 L_\odot}{4\pi(10\text{ pc})^2} = \frac{78.7 \times 3.83 \times 10^{26}}{4\pi \times (10 \times 3.09 \times 10^{16})^2} = 2.51 \times 10^{-8}\text{ W m}^{-2}$$

Alternative method using the apparent magnitude and brightness of the Sun:

$$m_\odot = \mathcal{M}_\odot + 5\log_{10}\left(\frac{1\text{ au in pc}}{10}\right) = 4.74 + 5\log_{10}\left(\left(\frac{1.50 \times 10^{11}}{3.09 \times 10^{16}}\right)/10\right) = -26.83$$

$$b_\odot = \frac{L_\odot}{4\pi(1\text{ au})^2} = \frac{3.83 \times 10^{26}}{4\pi(1.50 \times 10^{11})^2} = 1355\text{ W m}^{-2}$$

$$\therefore \frac{b_0}{b_\odot} = 10^{-0.4(0 - m_\odot)} \therefore b_0 = 10^{-0.4 \times 26.83} \times 1355 = 2.51 \times 10^{-8}\text{ W m}^{-2}$$

4. The star Bellatrix is in the constellation of Orion. If the star has a parallax of 13.0 mas and an apparent brightness of 5.56×10^{-9} W m^{-2} then calculate its
   a. Apparent magnitude
   b. Absolute magnitude
   c. Luminosity (in $L_\odot$)

Part a)
Using our result from earlier for the brightness of a zero-magnitude star:

$$m = -2.5\log_{10}\left(\frac{b}{b_0}\right) = -2.5\log_{10}\left(\frac{5.56 \times 10^{-9}}{2.51 \times 10^{-8}}\right) = 1.64$$

Part b)

$$d = \frac{1}{p} = \frac{1}{13.0 \times 10^{-3}} = 76.9\text{ pc}$$

$$\mathcal{M} = m - 5\log_{10}\left(\frac{d}{10}\right) = 1.64 - 5\log_{10}\left(\frac{76.9}{10}\right) = -2.79$$

Part c)

$$\mathcal{M} - \mathcal{M}_{\odot} = -2.5\log_{10}\left(\frac{L}{L_{\odot}}\right) \therefore L = 10^{\frac{-2.79-4.47}{-2.5}} L_{\odot} = 1030\, L_{\odot}$$

Alternative method using the given brightness and calculated distance

$$b = \frac{L}{4\pi d^2}$$

$$\therefore L = 4\pi d^2 b = 4\pi \times (76.9 \times 3.09 \times 10^{16})^2 \times 5.56 \times 10^{-9} = 3.95 \times 10^{29}\, W = 1030\, L_{\odot}$$