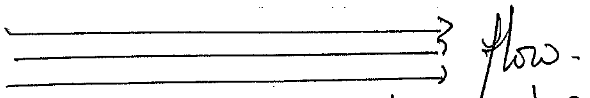
Round 2 Jan 2016

Que 1



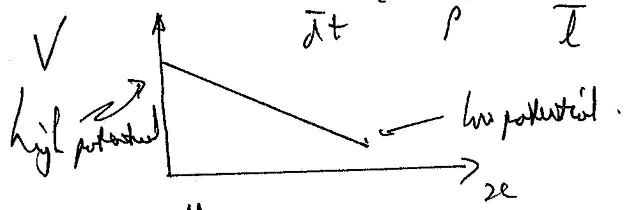
(a)(i)

Heat	rate	temperature gradient
Charge	$\frac{d}{dt}$	potential gradient.

$$\frac{dQ}{dt} = -\lambda . A . \frac{\Delta T}{\Delta x}$$

$$I = V/R \implies \frac{dQ}{dt} = \frac{A}{\rho l} . (V_1 - V_2) \text{ or potential difference}$$

so $$\frac{dQ}{dt} = \frac{1}{\rho} A \frac{\Delta V}{l}$$

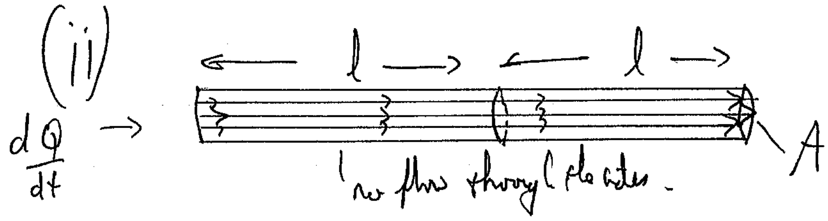


current flow in +ve x direction.

$$\therefore \quad \frac{dQ}{dt} = -\frac{1}{\rho} A . \frac{dV}{dx}$$

ρ is the resistivity. $\frac{dV}{dx}$ is the potential gradient.

(ii)



$$\Delta T = \Delta T_1 + \Delta T_2 \quad (\text{temperatures add})$$

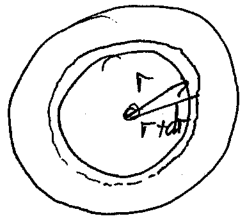
so $$\frac{dQ}{dt} \frac{2l}{\lambda A} = \frac{dQ}{dt} . \frac{l}{\lambda_1 A} + \frac{dQ}{dt} . \frac{l}{\lambda_2 A}$$

(for the whole length.

So that $$\frac{2}{\lambda} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$$

Qu 1 cont.

(b)



Current flows through a thin "circumference" of thickness, dr, area $2\pi r \times t$ and resistivity ρ.

$$\therefore \quad dR = \rho \frac{dr}{2\pi r t}$$

$$\text{So} \quad \int_0^R dR' = \frac{\rho}{2\pi t} \int_{r_0}^{r} \frac{dr'}{r'}$$

$$R = \frac{\rho}{2\pi t} \ln\left(\frac{r}{r_0}\right)$$

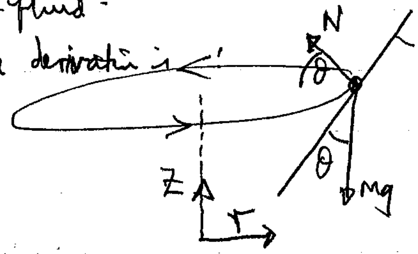
(c)

- The wavefronts can be seen to be curved and becoming slightly wider — an effect of diffraction
- There is a small acceleration of the water down the slope as the wavefronts become a little further apart.
- The water is very shallow, so its speed is limited by friction with the ground. It can be seen to be ~~the~~ shallow enough that the texture of the road shows in the ripples in the water surface (bottom left of Fig 1-2).
- The limited increase of speed ensures that the wavefronts do not spread out much. But there is a noticeable pile-up at each wavefront - as the water deepens slightly at a wavefront, the surface water moves over the layer close to the ground, giving a rolling breaker ~~eff~~ appearance at the wavefronts. This appears to maintain the wavefronts as it travels down the slope, so that they do not overtake each other and merge together.
- There must be a slight dip in the road as the water flows towards the centre. This also limits the diffraction and may cause the curvature of the wavefront, as the deeper water travels faster.

Qu 3

(a) There are several ways to derive the shape of the surface by considering elements of the fluid.

A simple derivation is



The particle of mass m is not in equilibrium as it is accelerating inwards. But it remains at the same height up the slope.

Resolving vertically : $N \sin\theta = mg$

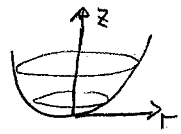
horizontally : $N \cos\theta = m r \omega^2$

$$\Rightarrow \tan\theta = \frac{mg}{mr\omega^2} = \frac{g}{r\omega^2}$$

With θ specified, $\tan\theta = \frac{dr}{dz}$

So $$\frac{dr}{dz} = \frac{g}{r\omega^2}$$

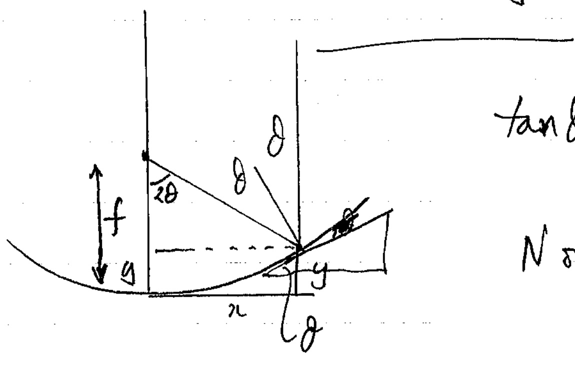
Integrating, $$\int r\omega^2 dr = \int g\, dz \Rightarrow z = \frac{r^2\omega^2}{2g} + c$$



$z = 0$ when $r = 0 \Rightarrow c = 0$

So $z = \frac{r^2\omega^2}{2g}$ (parabola).

(b)



$$\tan\theta = \frac{dy}{dx} = \frac{d}{dx}(ax^2) = 2ax$$

Now trigonometry!

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = (2\cos^2\theta - 1)$

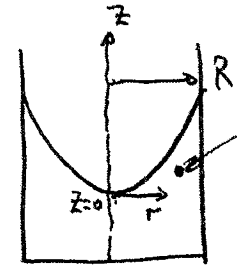
$$\therefore \tan 2\theta = \frac{2\sin\theta . \cos\theta}{2\cos^2\theta - 1}$$

$$= \frac{2\tan\theta}{2 - \frac{1}{\cos^2\theta}} = \frac{2\tan\theta}{1 + (1 - \frac{1}{\cos^2\theta})}$$

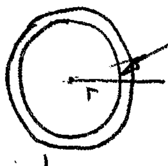
$$= \frac{2\tan\theta}{1 - \tan^2\theta} \quad \text{using } \tan\theta = 2ax$$

$$= \frac{4ax}{1 - 4a^2x^2}$$

Alternatives for parabolic surface of a rotating liquid mirror of mercury



consider a point in the fluid which has an element of mass given by its small dimensions.

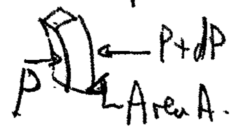


$dm = r\,d\theta . dr . dZ . \rho$

The element of mass is not in equilibrium since it is being accelerated.

The shape may be static but it is not an equilibrium shape.

From above:



$A\,dP = dF \Rightarrow dF = dm\,r.\omega^2$

$dP = \frac{dF}{A} = \frac{r\,d\theta . dr\,dz . \rho . r\omega^2}{r\,d\theta\,dz}$

$dP = \rho\omega^2 r\,dr$

$P = \frac{\rho\omega^2 r^2}{2} + c \quad : r=0, P=0 \Rightarrow c=0$

This assumes that $v = r\dot{\theta}$, i.e. the fluid flow is zero. The fluid rotates as a single unit.

(A) At a region in the fluid, B, the pressures are:

$$\rho g(z - z_B) \downarrow, \quad \rho\frac{r^2\omega^2}{2} \rightarrow \square \leftarrow \rho(r+dr)^2\frac{\omega^2}{2}, \quad \uparrow \rho g(z - z_B + dz)$$

$\rho g(z - z_B + dz) = \rho(r+dr)^2\frac{\omega^2}{2}$

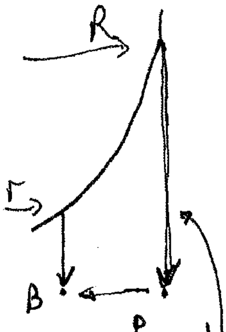
and $\rho r^2\frac{\omega^2}{2} = \rho g(z - z_B)$

so $\rho g\,dz = \rho r\,dr . \omega^2$

$\therefore g\,dz = \omega^2 r\,dr$

$Z = \frac{\omega^2 r^2}{2g} + c \qquad z=0, r=0 \Rightarrow c=0$

(B)



this creates pressure provides this pressure

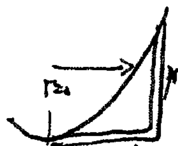
$\rho g(z_R - z_B - (z - z_B)) = \rho\frac{R^2\omega^2}{2} - \rho\frac{r^2\omega^2}{2}$

$(z_R - z) = \frac{\omega^2}{2g}(R^2 - r^2)$ at $z=0, r=0$

$z_R = \frac{\omega^2 R^2}{2g}$

so $z = \frac{\omega^2 r^2}{2g}$

(C)



pressure at bottom of the pressure to maintain fluid element at radius, r.

$\rho g z = \rho\frac{\omega^2 r^2}{2}$

$z = \frac{\omega^2 r^2}{2g}$

Qu 3 cont.

(c) From the diagram we can see that $\tan 2\theta = \frac{x}{f-y}$

To show that f is fixed, substitute for $y = ax^2$

Then $\tan 2\theta = \frac{x}{f-ax^2} = \frac{4ax}{4af-4a^2x^2}$

and so the term $4af = 1$

if $f = \frac{1}{4a}$ which is constant for all paraxial rays.

(d) From (a) we have $z = \frac{\omega^2}{2g} \cdot r^2$ and $y = ax^2$

so that $f = \frac{1}{4\,\omega^2/2g} = \frac{g}{2\omega^2}$

$\theta_{min} = 1.22\frac{\lambda}{D}$ using the Rayleigh criterion.

The resolution has to be 40 marcsec for red light (it will be better than this for blue).

$\therefore\ D = \frac{1.22\lambda}{\theta_{min}} = \frac{1.22 \times 700 \times 10^{-9}}{40 \times 10^{-3} \times \frac{1}{3600} \times \frac{\pi}{180}}$

$= \underline{4.40\ m}$

[and $f = \frac{9.81}{2 \times \left(\frac{8.5 \times 2\pi}{60}\right)^2} = \underline{6.19 m}$]

$dV = 2\pi x\, dx \cdot y$

$= 2\pi\, x\, dx \cdot ax^2$

$= 2\pi a\, x^3\, dx$

$\underline{W = 0.890\ rad\,s^{-1}}$

$V = \int_0^{D/2} 2\pi a\, x^3\, dx = 2\pi a \frac{x^4}{4}\Big|_0^{D/2}$

$= \frac{2\pi a}{4}\frac{D^4}{2^4} = \underline{\frac{\pi a\, D^4}{32}}$

and as, $a = \frac{1}{4f}$ $V = \frac{\pi\, D^4}{32 f . 4}$

$= \underline{1.49\ m^3}$

Qu 3 cont.

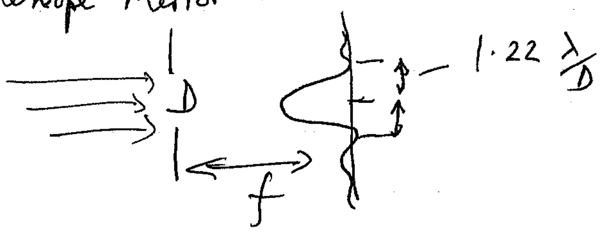
(e) Surface area $\sim \pi \frac{D^2}{4} = 15.2\,m^2$

Rate of loss $= \underline{0.77\ kg/week.}$

Rate is $\frac{0.77}{13600} \div 1.49 \times 100$ % per week.

$= \underline{3.8 \times 10^{-3}\ \%\ \text{per week.}}$

(f) The image of the almost point source (the star) has a circular diffraction pattern in the focal plane of the telescope mirror.



$$r_{diff} = f \times 1.22\frac{\lambda}{D}$$

$$\therefore \text{ the intensity factor } = \frac{\pi(D/2)^2}{\pi r_{diff}^2} = \frac{D^2}{4 r_{diff}^2}$$

$$= \frac{D^4}{4f^2\, 1.22^2\, \lambda^2}$$

$$= \frac{4.40^2}{4 \times 6.19^2 \times 1.22^2 \times 700^2 \times 10^{-18}}$$

$$= \underline{3.4 \times 10^{12}}$$

Intensity = Power/area $= 3.4 \times 10^{12} \times 8.0 \times 10^{-14}\ Wm^{-2}$

$= \underline{0.27\ Wm^{-2}}$

averaged over the central maximum area.

Question 4. (a)

$PV^{\gamma} = \text{const} = k$

(Adiabatic) W.D on a gas is

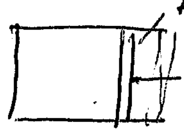
$$= -\int P\,dV = -k\int_{V_0}^{V_f} \frac{dV}{V^{\gamma}}$$

$$= -k \left. \frac{V^{-\gamma+1}}{-\gamma+1} \right|_{V_0}^{V_f}$$

$$= -k \frac{\left(V_f^{1-\gamma} - V_0^{1-\gamma}\right)}{1-\gamma}$$

$$= k \frac{\left(V_f^{1-\gamma} - V_0^{1-\gamma}\right)}{\gamma - 1}$$

ΔV



WD by gas is $P\Delta V$

$k = P_0 V_0^{\gamma} = P_f V_f^{\gamma}$

(Adiabatic) WD on a gas $= \dfrac{(P_f V_f - P_0 V_0)}{(\gamma - 1)}$.

WD on the gas on the RHS. is given by

$$WD = \frac{(P_f V_f - P_0 V_0)}{(\gamma - 1)}$$

with $P_f = \frac{27}{8} P_0$

$\gamma = 1.5$

and V_f given by

$P_0 V_0^{\gamma} = P_f V_f^{\gamma}$

So that, $\cancel{P_0} V_0^{\gamma} = \frac{27}{8} \cancel{P_0} \cdot V_f^{\gamma}$

Hence, $\left(\frac{V_0}{V_f}\right)^{1.5} = \left(\frac{27}{8}\right)$

$\frac{V_0}{V_f} = \frac{9}{4}$

$V_f = \frac{4}{9} V_0$

i.e. $WD = \dfrac{\frac{27}{8} \cdot \frac{4}{9} P_0 V_0 - P_0 V_0}{(1.5 - 1)}$

$= P_0 V_0 \dfrac{\left(\frac{3}{2} - 1\right)}{0.5}$

$= \boxed{P_0 V_0} = \boxed{nRT_0}$

{and $P_0 V_0 = nRT_0$}

This result can also be obtained by knowing T_f from part (b) ($T_f = \frac{3}{2} T_0$). However care must be taken as the molecules are not point particles so $U_0 = 2NkT_0$, not $\frac{3}{2} kT_0$.

With $\gamma = \frac{3}{2}$, $\gamma = (1 + \frac{2}{f})$ with f the degrees of freedom. Here, $f = 4$ (3 translational and 1 other).

So $U_0 = 4 \cdot \frac{1}{2} NkT_0$ and with $T_f = \frac{3}{2} T_0$

$U_f = 4 \cdot \frac{1}{2} Nk\left(\frac{3}{2} T_0\right)$ So $\Delta U = 2Nk \cdot \frac{3}{2} T_0 - 2NkT_0 = NkT_0 = nRT_0$

Question ~~4~~ cont. ...

(b) $$PV^{\gamma} = \text{const.} \quad (1)$$

and $\frac{PV}{T} = \text{const}$ so that $\frac{P^{\gamma}V^{\gamma}}{T^{\gamma}} = \text{const.} \quad (2)$

Hence, dividing (1) by (2)

We obtain $P^{\gamma - 1} . T = \text{const}$

Now we can substitute.

$$P_0^{\frac{2}{3}-1} \times T_0 = \left(\frac{27}{8} P_0\right)^{\frac{2}{3}-1} \times T_f$$

$$\frac{T_0}{P_0^{\frac{1}{3}}} = \left(\frac{27}{8} P_0\right)^{-\frac{1}{3}} \times T_f$$

$$\frac{T_0}{\cancel{P_0^{\frac{1}{3}}}} = \frac{2}{3 \cancel{P_0^{\frac{1}{3}}}} T_f$$

$$T_0 = \frac{2}{3} T_f$$

so $$T_f = \frac{3}{2} T_0$$

Alternatively

$$dU = nC_V . \Delta T$$

even though the volume is not constant here. The internal energy increase is due to the temperature rise and the molar specific thermal capacity when no work is done on or by the gas.

and we know from part (a) that

dU = the WD on the gas in an adiabatic compression in which no heat is exchanged with the surroundings.

$$\therefore \quad nC_V(T_f - T_0) = nRT_0$$

$$T_f - T_0 = \frac{R}{C_V} . T_0$$

$$T_f - T_0 = (\gamma - 1) T_0$$

$$T_f = \gamma T_0 - T_0 + T_0$$

$$T_f = \frac{3T_0}{2} \quad \text{or} \quad T_f = \left(1 + \frac{R}{C_V}\right) T_0$$

Now this step needs Mayer's equation

$$C_P - C_V = R$$

so that $\frac{C_P}{C_V} - 1 = \frac{R}{C_V}$

$$\gamma - 1 = \frac{R}{C_V}$$

Qu. ~~B~~ cont...

(c) Pressure is $\frac{27}{8} P_0$ on LHS, same as on RHS.

And the total volume of $2V_0$ is $V_{LHS} + V_{RHS}$

and we know from part (a) that $V_{RHS} = \frac{4}{9} V_0$

Hence $V_{LHS} = 2V_0 - \frac{4}{9} V_0$

$$V_{LHS} = \frac{14}{9} V_0$$

~~or alternatively~~ Details: $P_0 V_0^{\gamma} = P_f V_f^{\gamma}$ for the RHS.

$$P_0 V_0^{\gamma} = \frac{27}{8} P_0 \times V_f^{\gamma}$$

$$\text{RHS} \longrightarrow V_f = \left(\frac{8}{27}\right)^{\frac{1}{\gamma}} V_0 = \frac{4}{9} V_0$$

~~Hence~~ so $\text{LHS} \longrightarrow V_f = \frac{14}{9} V_0$

Using the ideal gas law.

for LHS,

$$\frac{P_0 V_0}{T_0} = \frac{P_f V_f}{T_f}$$

$$\frac{\cancel{P_0 V_0}}{T_0} = \frac{\frac{27}{8}\cancel{P_0} \times \frac{14}{9}\cancel{V_0}}{T_f}$$

$$T_f = \frac{27}{8} \cdot \frac{14}{9} T_0$$

$$T_f = \frac{21}{4} T_0$$

(d) The gas on the LHS has thermal energy supplied, so it gains internal energy and its temperature rises, but it also does work in expanding.
Remember, as thermal energy is supplied, this is not an adiabatic change, so the adiabatic formula for WD does not apply on the LHS.

Gain in internal energy, $\Delta U = n C_V (T_f - T_0)$

$$= n C_V . \left(\frac{21}{4} T_0 - T_0\right)$$

$$= \underline{n . C_V . \frac{17}{4} . T_0}$$

[or if we use $C_p - C_V = R$ so that $\gamma - 1 = \frac{R}{C_V}$ then

$$\Delta U = n R \frac{17}{2} T_0$$

$$= \underline{\frac{17}{2} P_0 V_0}$$]

Since the WD on the RHS is $P_0 V_0$ or $n R T_0$ from part (a) this is the Work done by the gas on the LHS.

$\therefore$ ΔQ = increase in thermal energy + work done by the gas in expanding

$$= \frac{17}{2} P_0 V_0 + P_0 V_0$$

$$= \underline{\underline{\frac{19}{2} P_0 V_0}} = \underline{\underline{\frac{19}{2} n R T_0}} = n C_V \frac{17}{4} T_0 + n R T_0$$

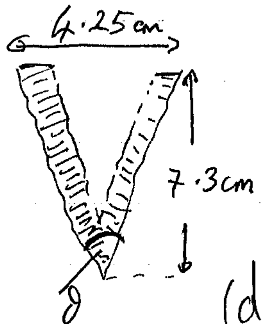
$$= \underline{\underline{n T_0 \left(\frac{17}{4} C_V + R\right)}}$$

Question 5

(a) Scale : on paper 10·2 cm corresponds to ~~7~~·8 cm on screen.
1 cm " " 0·765 cm on screen

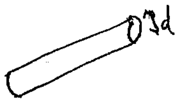
(i) Smallest fringe spacing : 24 fringes ~ 5·2 cm is 0·166 cm on screen

(ii) Largest spacing 4 fringes (a diagonal) in ~~4~~·2 cm is 0·803 cm on screen
(and there are 5 small fringes in 1 large fringe)



$\theta = 32{\cdot}5^\circ = \underline{\underline{33^\circ}}$

(d) The wire thickness produces the wide spaced fringe pattern.



$$\frac{\lambda}{\text{wire diameter}} = \frac{\text{fringe spacing}}{\text{distance to screen}}$$

$$\frac{\lambda}{d} = \frac{W}{D}$$

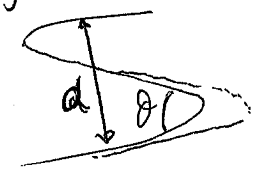
$$\therefore\ d = \frac{\lambda D}{W} = \frac{633\times10^{-9}\times 4{\cdot}2}{0{\cdot}803\times10^{-2}}$$

$$= 3{\cdot}3\times10^{-4}\ \text{m}$$

$$= 0{\cdot}33\ \text{mm}.$$

N.B. The diffraction pattern does not correspond to the dimensions of the spring illustrated.

(iii)

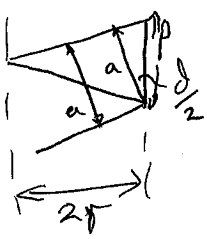


$$a = \frac{\lambda D}{W} = \frac{633\times10^{-9}\times 4{\cdot}2}{0{\cdot}166\times10^{-2}}$$

$$= 1{\cdot}60\times10^{-3}\ \text{m}$$

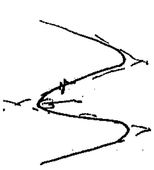
$$= 1{\cdot}6\ \text{mm}$$

(iv)



pitch, $p = \dfrac{a}{\cos\theta/2}$ $= 1{\cdot}67$
$= \underline{1{\cdot}7\ \text{mm}}$

(v)



projected spring is a sine curve

$$\frac{2\pi r}{p} = \tan\left(90 - \frac{\theta}{2}\right)$$

so, $\underline{2r = 1{\cdot}79\ \text{mm}}$, $\underline{r = 0{\cdot}897\ \text{mm}}$

Qu. 5 cont.

(b) (i) Scale 9.1 cm on paper is 9.4 cm on screen

4.5 cm for 10 fringes

$\therefore$ fringe on screen is $W = 0.465$ cm

(ii) $a = \frac{\lambda D}{W} = \frac{0.15 \times 10^{-9} \times 9.0 \times 10^{-2}}{0.465 \times 10^{-2}}$

$= 2.9 \times 10^{-9}$ m

$= \underline{2.9 \text{ nm}}$

(iii) $\theta = 84°$

(iv) pitch, $p = \frac{a}{\cos\frac{\theta}{2}} = \frac{2.9}{\cos 42} = \underline{3.9 \text{ nm}}$

(v) radius; $\frac{2\pi r}{p} = \tan(90 - \frac{\theta}{2})$

$\underline{\underline{r = 0.69 \text{ nm}}}$