3 Rotation

Rotational motion is all around us [groan] – from the acts of subatomic particles, to the motion of galaxies. Calculations involving rotations are no harder than linear mechanics; however the quantities we shall be talking about will be unfamiliar at first. Having already studied linear mechanics, you will be at a tremendous advantage, since we shall find that each ‘rule’ in linear mechanics has its rotational equivalent.

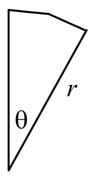
3.1 *Angle*

In linear mechanics, the most fundamental measurement is the position of the particle. The equivalent base of all rotational analysis is angle: the question “How far has the car moved?” being exchanged for “How far has the wheel gone round?” – a question which can only be answered by giving an angle. In mechanics, the radian is used for measuring angles. While you may be more familiar with the degree, the radian has many advantages.

We shall start, then by defining what we mean by a radian. Consider a sector of a circle, as in the diagram; and let the circle have a radius *r*. The length of the arc, that is the curved line in the sector, is clearly related to the angle. If the angle were made twice as large, the arc length would also double.

Can we use arc length to measure the angle? Not as it stands, since we haven’t taken into account the radius of the circle. Even for a fixed angle (say 30°), the arc will be longer on a larger circle. We therefore define the angle (in radians) as the arc length divided by the circle radius. Alternatively you might say that the angle in radians is equal to the length of the arc of a unit circle (that is a circle of 1m radius) that is cut by the angle.

Arc length = $r\theta$ if θ is measured in radians

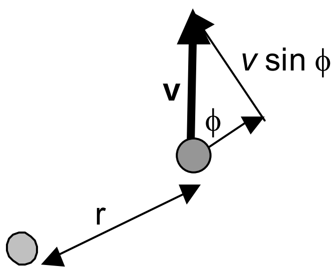


Notice one simplification that this brings. If a wheel, of radius *R*, rolls a distance *d* along a road, the angle the wheel has turned through is given by *d/R* in radians. Were you to calculate the angle in degrees, there would be nasty factors of 180 and π in the answer.

Before getting too involved with radians, however, we must work out a conversion factor so that angles in degrees can be expressed in radians. To do this, remember that a full circle (360°) has a circumference or arc length of $2\pi r$. So 360°=2π rad. Therefore, 1 radian is equivalent to (360/2π)° = (180/π)°.

3.2 *Angular Velocity*

Having discussed angle as the rotational equivalent of position, we now turn our attention to speed. In linear work, speeds are given in metres per second – the distance moved in unit time. For rotation, we speak of 'angular velocity', which tells us how fast something is spinning: how many radians it turns through in one second. The angular velocity can also be thought of as the derivative of angle with respect to time, and as such is sometimes written as $\dot{\theta}$, however more commonly the Greek letter ω is used, and the dot is avoided. To check your understanding of this, try and show that 1 rpm (revolution per minute) is equivalent to π/30 rad/s, while one cycle per second is equivalent to 2π rad/s.



Now remember the definition of angle in radians, and that the distance moved by a point on the rim of a wheel will move a distance $s = r\theta$ when the wheel rotates by an angle θ. The speed of the point will therefore be given by $u = \mathrm{d}s/\mathrm{d}t = r\,\mathrm{d}\theta/\mathrm{d}t = r\omega$.

For a point that is not fixed to the wheel, the situation is a little more complex. Suppose that the point has a velocity **v**, which makes an angle ϕ to the radius (as in the figure above). We then separate **v** into two components, one radial ($v \cos \phi$) and one rotational ($v \sin \phi$). Clearly the latter is the only one that contributes to the angular velocity, and therefore in this more general case, $v \sin \phi = r\omega$.

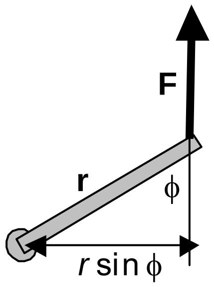
3.3 *Angular acceleration*

It should come as no surprise that the angular acceleration is the time derivative of ω, and represents the change in angular velocity (in rad/s) divided by the time taken for the change (in s). It is measured in rad/s^2, and denoted by α or $\dot{\omega}$ or $\ddot{\theta}$. For an object fastened to the rim of a wheel, the 'actual' acceleration round the rim (a) will be given by $a=\mathrm{d}u/\mathrm{d}t = r\,\mathrm{d}\omega/\mathrm{d}t = r\alpha$, while for an object not fastened, we have $a \sin \phi = r\alpha$.[8]

3.4 *Torque – Angular Force*

Before we can start 'doing mechanics' with angles, we need to consider the rotational equivalent of force – the amount of twist. Often a twist can be applied to a system by a linear force, and this gives us a 'way in' to the analysis. We say that the strength of the twist is called the 'moment' of the force, and is equal to the size of the force multiplied by the distance from pivot to the point where the force acts. A complication

[8] Here we are not including the centripetal acceleration which is directed towards the centre of the rotation.



arises if the force is not tangential – clearly a force acting along the radius of a wheel will not turn it – and so our simple 'moment' equation needs modifying.[9]

There are two ways of proceeding, and they yield the same answer. Suppose the force F makes an angle ϕ with the radius. We can break this down into two components – one of magnitude $F \cos \phi$, which is radial and does no turning; and the other, tangential component (which does contribute to the turning) of magnitude $F \sin \phi$. The moment or torque only includes the relevant component, and so the torque is given by $C = Fr \sin \phi$.

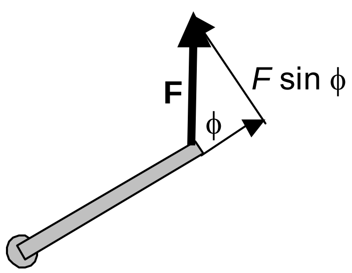
The alternative way of viewing the situation is not to measure the distance from the centre to the *point at which the force is applied*. Instead, we draw the force as a long line, and to take the distance as the *perpendicular* distance from force line to centre. The diagram shows that this new distance is given by $r \sin \phi$, and since the force here is completely tangential, we may write the moment or torque as the product of the full force and this perpendicular distance – i.e. $C = F\, r \sin \phi$, as before.

3.5 *Moment of Inertia – Angular Mass*

Of the three base quantities of motion, namely distance, mass and time, only time may be used with impunity in rotational problems. We now have an angular equivalent for distance (namely angle), so the next task is to determine an angular equivalent for mass.

This can be done by analogy with linear mechanics, where the mass of an object in kilograms can be determined by pushing an object, and calculating the ratio of the applied force to the acceleration it caused: $m = F/a$. Given that we now have angular equivalents for force and acceleration, we can use these to find out the 'angular mass'.

Think about a ball of mass m fixed to the rim of a wheel that is accelerating with angular acceleration α. We shall ignore the mass of the wheel itself for now. Now let us push the mass round the wheel with a force F. Therefore we calculate the 'angular mass' I by



[9] Why force × radius? We can use a virtual work argument (as in section 1.1.1.4) to help us. Suppose a tangential force F is applied at radius r. When the object moves round by angle θ, it moves a distance $d = \theta r$, and the work done by the force = $Fd = F\theta r = Fr \times \theta =$ angular force × angular distance. Now since energy must be the same sort of thing with rotational motion as linear, the rotational equivalent of force must be Fr.

$$I = \frac{C}{\alpha} = \frac{rF\sin\phi}{\left(a\sin\phi/r\right)} = r^2\ \frac{F}{a} = r^2\ m$$

where we have used the fact that the mass *m* will be the ratio of the force *F* to the linear acceleration *a*, as dictated by Newton's Second Law. This formula can also be used for solid objects, however in this case, the radius *r* will be the perpendicular distance from the mass to the axis. The total 'angular mass' of the object is calculated by adding up the $I = m\ r^2$ from each of the points it is made from.

Usually this 'angular mass' is called the **moment of inertia** of the object. Notice that it doesn't just depend on the mass, but also on the distance from the point to the centre. Therefore the moment of inertia of an object *depends on the axis* it is spun round.

An object may have a high angular inertia, therefore, for two reasons. Either it is heavy in its own right; or for a lighter object, the mass is a long way from the axis.

3.6 *Angular Momentum*

In linear motion, we make frequent use of the 'momentum' of objects. The momentum is given by mass × velocity, and changes when a force is applied to the object. The force applied, is in fact the time derivative of the momentum (provided that the mass doesn't change). Frequent use is made of the fact that total momentum is conserved in collisions, provided that there is no *external* force acting.

It would be useful to find a similar 'thing' for angular motion. The most sensible starting guess is to try 'angular mass' × angular velocity. We shall call this the angular momentum, and give it the symbol $L = I\ \omega$. Let us now investigate how the angular momentum changes when a torque is applied. For the moment, assume that *I* remains constant.

$$\frac{dL}{dt} = \frac{d}{dt} I\omega = I\frac{d\omega}{dt} = I\alpha = C$$

Thus we see that, like in linear motion, the time derivative of angular momentum is 'angular force' or torque. Two of the important facts that stem from this statement are:

1. If there is no torque *C*, the angular momentum will not change. Notice that radial forces have $C = 0$, and therefore will not change the angular momentum. This result may seem unimportant – but think of the planets in their orbits round the Sun. The tremendous force exerted on them by the Sun's gravity is *radial*, and therefore does not change their angular momenta even a smidgen. We can therefore calculate the velocity of planets at different parts of their orbits using the fact that the angular momentum will remain the same. This principle also holds when

scientists calculate the path of space probes sent out to investigate the Solar System.

2. The calculation above assumes that the moment of inertia *I* of the object remains the same. This seems sensible, after all, in a linear collision, the instantaneous change of a single object's mass would be bizarre[10], and therefore we don't need to guard against the possibility of a change in mass when we write $F = \mathrm{d}p/\mathrm{d}t$.

In the case of angular motion, this situation is different. The moment of inertia can be changed, simply by rearranging the mass of the object closer to the axis. Clearly there is no *external* torque in doing this, so we should expect the angular momentum to stay the same. But if the mass has been moved closer to the axis, *I* will have got smaller. Therefore ω must have got bigger. The object will now be spinning faster! This is what happens when a spinning ice dancer brings in her/his arms – and the corresponding increase in revs. per minute is well known to ice enthusiasts and TV viewers alike.

To take an example, suppose that all the masses were moved twice as close to the axis. The value of *r* would halve, so *I* would be quartered. We should therefore expect ω to get four times larger. This is in fact what happens.

3.7 *Angular momentum of a single mass moving in a straight line*

If we wished to calculate the angular momentum of a planet in its orbit round the Sun, we need to know how *L* is related to the linear speed *v*. This is what we will now work out.

Using the same ideas as in figure 2, the velocity **v** will have both radial and 'rotational' components. The rotational component will be equal to $v \sin \phi$, while the radial component cannot contribute to the angular momentum. It is the rotational component that corresponds to the speed of a mass fixed to the rim of a wheel, and as such is equal to radius × angular velocity. Thus $v \sin \phi = r\,\omega$. So the angular momentum

$$L = I\omega = mr^2 \times \frac{v \sin \phi}{r} = mvr \sin \phi$$

[10] Two cautions. Firstly, in a rocket, the mass of the rocket does decrease as the burnt fuel is chucked out the back, however the total mass does not change. Therefore $F=\mathrm{d}p/\mathrm{d}t=ma$ still works, we just need to be careful that the force *F* acts on (and only on) the stuff included in the mass *m*. A complication *does* arise when objects start travelling at a good fraction of the speed of light – but this is dealt with in the section on Special Relativity.

is given by the product of the mass, the radius and the rotational (or tangential) component of the velocity.

For an object on a *straight line path*, this can also be stated (using figure 3) as the mass × speed × distance of closest approach to centre.

3.8 Rotational Kinetic Energy

Lastly, we come to the calculation of the rotational kinetic energy. We may calculate this by adding up the linear kinetic energies of the parts of the object as the spin round the axis. Notice that in this calculation, as the objects are purely rotating, we shall assume $\phi = \pi/2$ – i.e. there is no radial motion.

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m(r\omega)^2 = \tfrac{1}{2}mr^2\omega^2 = \tfrac{1}{2}I\omega^2$$

We see that the kinetic energy is given by half the angular mass × angular velocity squared – which is a direct equivalent with the half mass × speed^2 of linear motion.

3.9 Summary of Quantities

Quantity	Symbol	Unit	Definition	Other equations
Angular velocity	ω	rad/s	$\omega = d\theta/dt$	$r\,\omega = v \sin\phi$
Angular acceleration	α	rad/s^2	$\alpha = d\omega/dt$	$r\,\alpha = a \sin\phi$
Torque	C	N m	$C = F\,r \sin\phi$	
Moment of inertia	I	kg m^2	$I = C / \alpha$	$I = m\,r^2$
Angular momentum	L	kg m^2 /s	$L = I\,\omega$	$L = m\,v\,r \sin\phi$
Rot. Kinetic Energy	K	J	$K = I\,\omega^2 / 2$	$K = \tfrac{1}{2}m\,(v \sin\phi)^2$

3.10 *Rotational mechanics with vectors*

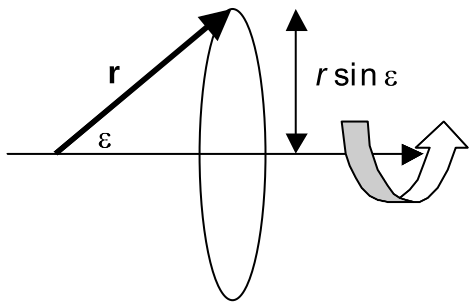
This section involves much more advanced mathematics, and you will be able to get by in Olympiad problems perfectly well without it. However, if you like vectors and matrices, read on...

So far we have just considered rotations in one plane – that of the paper. In general, of course, rotations can occur about any axis, and to describe this three dimensional situation, we use vectors. With velocity **v**, momentum **p** and force **F**, there is an obvious direction – the direction of motion, or the direction of the 'push'. With rotation, the 'direction' is less clear.

Imagine a clock face on this paper, with the minute hand rotating clockwise. What direction do we associate with this motion? Up towards 12 o'clock because the hand sometimes points that way? Towards 3 o'clock because the hand sometimes points that way? Both are equally ridiculous. In fact the only way of choosing a direction that will always apply is to assign the rotation 'direction' *perpendicular* to the clock face – the direction in which the hands *never* point.

This has not resolved our difficulty completely. Should the arrow point upwards out of the paper, or down into it? After thought we realise that one should be used for clockwise and one for anti-clockwise motion, but which way? There is no way of proceeding based on logic – we just have to accept a convention. The custom is to say that for a clockwise rotation, the 'direction' is down away from us, and for anticlockwise rotation, the direction is up towards us.

Various aides-memoire have been presented for this – my favourite is to consider a screw. When turned clockwise it moves away from you: when turned anticlockwise it moves towards you. For this reason the convention is sometimes called the 'right hand screw rule'.



With this convention established, we can now use vectors for angular velocity ω, angular momentum **L**, and torque **C**. Kinetic energy, like in linear motion, is a scalar and therefore needs no further attention. The moment of inertia *I* is more complex, and we shall come to that later.

Let us consider the angular velocity first. If we already know ω and **r**, what is **v**, assuming that only rotational velocities are allowed? Remembering that w must point along the axis of the rotation, we may draw the diagram above, which shows that the radius of the circle that our particle actually traces out is $r \sin \varepsilon$ where ε is the angle between **r** and ω.[11] This factor of $\sin \varepsilon$ did not arise before in this way, since our motion was restricted to the plane which contained the centre point, and thus $\varepsilon = \pi/2$ for all our 2-dimensional work. Therefore the velocity is equal to w multiplied by the radius of the circle traced out, i.e. $v = \omega r \sin \varepsilon$. This may be put on a solid mathematical foundation using the vector cross product namely $\mathbf{v} = \omega\times\mathbf{r}$. This is our first vector identity for rotational motion.

By a similar method, we may analyse the acceleration. We come to the corresponding conclusion $\mathbf{a} = \alpha \times \mathbf{r}$.[12]

Next we tackle torque. Noting our direction convention, and our earlier equation $C = F r \sin \phi$, we set $\mathbf{C} = \mathbf{r} \times \mathbf{F}$. Similarly, from $L = (mv) r \sin \phi = p r \sin \phi$, we set $\mathbf{L} = \mathbf{r} \times \mathbf{p}$.

With these three vector equations we may get to work. Firstly, notice:

$$\frac{d}{dt}\mathbf{L} = \frac{d}{dt}(\mathbf{r}\times\mathbf{p}) = \mathbf{v}\times\mathbf{p} + \mathbf{r}\times\frac{d}{dt}\mathbf{p} = \mathbf{0} + \mathbf{r}\times\frac{d}{dt}(m\mathbf{v}) = \mathbf{r}\times m\mathbf{a} = \mathbf{r}\times\mathbf{F} = \mathbf{C}$$

The time derivative of angular momentum is the torque, as before. Notice too that the (**v×p**) term disappears since **p** has the same direction as **v**, and the vector cross product of two parallel vectors is zero.

3.10.1.1 General Moment of Inertia

Our next task is to work out the moment of inertia. This can be more complex, since it is not a vector. Previously we defined I by the relationships $C = I\,\alpha$, and also used the expression $L = I\,\omega$. Now that **C**, α, **L** and ω are vectors, we conclude that I must be a matrix, since a vector is made when I is multiplied by the vectors α or ω. Our aim is to find the matrix that does the job.

For this, we use our vector equations $\mathbf{v} = \omega \times \mathbf{r}$ and $\mathbf{C} = \mathbf{r} \times \mathbf{F}$, we let the components of **r** be (x,y,z), and we also use the mathematical result that for any three vectors **A**, **B** and **C**, $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\bullet\mathbf{C})\mathbf{B} - (\mathbf{A}\bullet\mathbf{B})\mathbf{C}$.

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[11] We use ε to represent the angle between **r** and ω, to distinguish it from the angle ϕ between **r** and **v**, which is of course a right angle for a strict rotation.

[12] This intentionally does not include the centripetal acceleration, as before. If you aim to calculate this **a** from the former equation $\mathbf{v} = \omega \times \mathbf{r}$, then you get $\mathbf{a} = d\mathbf{v}/dt = d(\omega\times\mathbf{r})/dt = \alpha\times\mathbf{r} + \omega\times\mathbf{v} = \alpha\times\mathbf{r} + \omega\times(\omega\times\mathbf{r}) = \alpha\times\mathbf{r} + \omega\,(\mathbf{r}.\omega) - \mathbf{r}\,\omega^2$. The final two terms in this equation deal with the centripetal acceleration. However in real situations, the centripetal force is usually provided by internal or reaction forces, so often problems are simplified by not including it.

$$\begin{aligned}
\mathbf{C} &= \mathbf{r}\times\mathbf{F} = m(\mathbf{r}\times\mathbf{a}) \\
&= m[\mathbf{r}\times(\boldsymbol{\alpha}\times\mathbf{r})] \\
&= mr^2\boldsymbol{\alpha} - m(\mathbf{r}\bullet\boldsymbol{\alpha})\mathbf{r} \\
&= m\begin{pmatrix} r^2-x^2 & -xy & -xz \\ -xy & r^2-y^2 & -yz \\ -xz & -yz & r^2-z^2 \end{pmatrix}\boldsymbol{\alpha} \\
&= m\begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix}\boldsymbol{\alpha}
\end{aligned}$$

This result looks horrible. However let us simplify matters by aligning our axes so that the z axis is the axis of the acceleration α. In other words $\boldsymbol{\alpha} = (0,0,\alpha)$. We now have

$$\mathbf{C} = m\begin{pmatrix} -xz \\ -yz \\ x^2+y^2 \end{pmatrix}\alpha$$

which is a little better. Notice that it is still pretty nasty in that the torque required to cause this z-rotation acceleration is not necessarily in the z-direction! Another consequence of this is that the angular momentum **L** is not necessarily parallel to the angular velocity ω. However for many objects, we rotate them about an axis of symmetry. In this case the xz and yz terms become zero when summed for all the masses in the object, and what we are left with is the mass multiplied by the distance from the axis to the masses (that is $x^2 + y^2$). Alternatively, for a flat object (called a lamina) which has no thickness in the z direction, the xz and yz terms are zero anyway, because z=0.

At this point, you are perfectly justified in saying 'yuk' and sticking to two-dimensional problems. However this result we have just looked at has interesting consequences. When a 3-d object has little symmetry, it can roll around in some very odd ways. Some of the asteroids and planetary moons in our Solar System are cases in point.

The moment of inertia can also be obtained from the rotational momentum, however, the form is identical to that worked out above from Newton's second law, as shown here.

$$\begin{aligned}
\mathbf{L} &= \mathbf{r}\times\mathbf{p} = m(\mathbf{r}\times\mathbf{v}) \\
&= m[\mathbf{r}\times(\boldsymbol{\omega}\times\mathbf{r})] \\
&= mr^2\boldsymbol{\omega} - m(\mathbf{r}\bullet\boldsymbol{\omega})\mathbf{r}
\end{aligned}$$

The calculation then proceeds as before.

3.10.1.2 General Kinetic Energy

Our final detail is kinetic energy. This can be calculated using **v** = ω × **r**, and the vector rule that $\mathbf{A}\bullet(\mathbf{B}\times\mathbf{C})=\mathbf{B}\bullet(\mathbf{C}\times\mathbf{A})$.

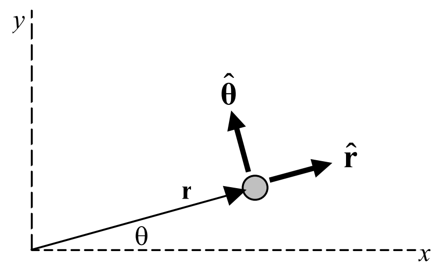
$$\begin{aligned}
K &= \tfrac{1}{2}mv^2 = \tfrac{1}{2}m(\mathbf{v}\bullet\mathbf{v}) \\
&= \tfrac{1}{2}m[\mathbf{v}\bullet(\boldsymbol{\omega}\times\mathbf{r})] \\
&= \tfrac{1}{2}m[\boldsymbol{\omega}\bullet(\mathbf{r}\times\mathbf{v})] \\
&= \tfrac{1}{2}\boldsymbol{\omega}\bullet(\mathbf{r}\times m\mathbf{v}) \\
&= \tfrac{1}{2}\boldsymbol{\omega}\bullet(\mathbf{r}\times\mathbf{p}) \\
&= \tfrac{1}{2}\boldsymbol{\omega}\bullet\mathbf{L} = \tfrac{1}{2}\boldsymbol{\omega}\bullet I\boldsymbol{\omega}
\end{aligned}$$

For the cases where I can be simplified, this reduces to the familiar form $K = I\,\omega^2/2$.

3.11 *Motion in Polar Co-ordinates*

When a system is rotating, it often makes sense to use polar co-ordinates. In other words, we characterise position by its distance from the centre of rotation (i.e. the radius *r*) and by the angle θ it has turned through. Conversion between these co-ordinates and our usual Cartesian (*x,y*) form are given by simple trigonometry:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \qquad (1)$$



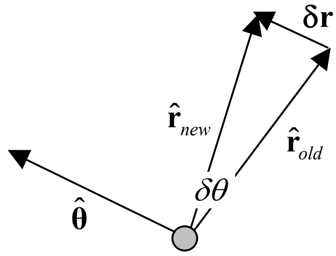
When analysing motion problems, though, there are complications if polar co-ordinates are used. These stem from the fact that the 'increasing *r*' and 'increasing θ' directions themselves depend on the value of θ, as we shall see. Let us start by defining the vector **r** to be the position of a particle relative to some convenient origin. The length of this vector *r* gives the distance from particle to origin. We define $\hat{\mathbf{r}}$ to be a unit vector parallel to **r**. Similarly, we define the vector $\hat{\boldsymbol{\theta}}$ to be a unit vector pointing in the direction the particle would have to go in order to increase θ while keeping *r* constant. Let us now evaluate the time derivative of **r** – in other words, let's find the velocity of the particle:

$$\frac{d}{dt}\mathbf{r} = \frac{d(r\hat{\mathbf{r}})}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} = \dot{r}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt}, \qquad (2)$$

where we have used the dot above a letter to mean 'time derivative of'. Now if the particle does not change its θ, then the direction $\hat{\mathbf{r}}$ will not change either, and we have a velocity given simply by $\dot{r}\hat{\mathbf{r}}$. We next consider the case when *r* doesn't change, and the particle goes in a circle around the origin. In this case, our formula would say that the velocity was $r\frac{d\hat{\mathbf{r}}}{dt}$. We know from section 3.2 that in this case, the speed is given by *r*ω, that is $r\dot{\theta}$, so the velocity will be $r\dot{\theta}\hat{\boldsymbol{\theta}}$. In order to make this agree with our equation for d**r**/dt, we would need to say that

$$\frac{d}{dt}\hat{\mathbf{r}} = \dot{\theta}\hat{\boldsymbol{\theta}}. \qquad (3)$$

Does this make sense? If you think about it for a moment, you should find that it does. Look at the diagram below. Here the angle θ has changed a small amount δθ. The old and new $\hat{\mathbf{r}}$ vectors are shown, and form two sides of an isosceles triangle, the angle between them being δθ. Given that the sides $\hat{\mathbf{r}}$ have length 1, the length of the third side is going to be approximately equal to δθ (with the approximation getting better the smaller δθ is). Notice also that the third side – the vector corresponding to $\hat{\mathbf{r}}_{new} - \hat{\mathbf{r}}_{old}$ is pointing in the direction of $\hat{\boldsymbol{\theta}}$. This allows us to justify statement (1).



In a similar way, we may show that

$$\frac{d}{dt}\hat{\boldsymbol{\theta}} = -\dot{\theta}\hat{\mathbf{r}}. \qquad (4)$$

Remembering that our velocity is given by

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}},$$

we may calculate the acceleration as

$$\begin{aligned}\mathbf{a} = \dot{\mathbf{v}} &= \ddot{r}\,\hat{\mathbf{r}} + \dot{r}\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\ddot{\theta}\,\hat{\boldsymbol{\theta}} + \dot{r}\frac{d\,\hat{\mathbf{r}}}{dt} + r\dot{\theta}\frac{d\,\hat{\boldsymbol{\theta}}}{dt} \\ &= \ddot{r}\,\hat{\mathbf{r}} + \dot{r}\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\ddot{\theta}\,\hat{\boldsymbol{\theta}} + \dot{r}\dot{\theta}\,\hat{\boldsymbol{\theta}} - r\dot{\theta}^2\hat{\mathbf{r}} \\ &= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\boldsymbol{\theta}}\end{aligned} \quad . \qquad (5)$$

Now suppose that a force acting on the particle (with mass m), had a radial component F_r, and a tangential component F_θ. We could then write

$$\begin{aligned}F_r &= m\left(\ddot{r} - r\dot{\theta}^2\right) \\ F_\theta &= m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\end{aligned} . \qquad (6)$$

There are many consequences of these equations for rotational motion. Here are three:

1. For an object to go round in a circle (that is *r* staying constant, so that $\dot{r} = \ddot{r} = 0$), we require a non-zero radial force $F_r = -mr\dot{\theta}^2$. The minus sign indicates that the force is to be in the opposite direction to **r**, in other words pointing towards the centre. This, of course, is the *centripetal* force needed to keep an object going around in a circle at constant speed.

2. If the force is purely radial (we call this a central force), like gravitational attraction, then $F_\theta = 0$. It follows that

$$\begin{aligned}0 &= mr\ddot{\theta} + 2m\dot{r}\dot{\theta} \\ &= mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}\,, \\ &= \frac{d}{dt}\left(mr^2\,\dot{\theta}\right)\end{aligned} \qquad (7)$$

   and accordingly the angular momentum $mr^2\dot{\theta} = mr^2\omega$ does not change. This ought to be no surprise, since we found in section 3.6 that angular momenta are only changed if there is a torque, and a radial force has zero torque.

3. One consequence of the conservation of angular momentum is the apparently odd behaviour of an object coming obliquely towards the centre (that is, it gets closer to the origin, but is not aimed to hit it). Since *r* decreases, ω must increase, and this is what happens – in fact the square term causes ω to quadruple when *r* halves.

   We can analyse this in terms of forces using (6): $F_\theta = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)$ when $F_\theta = 0$. Since *r* is decreasing, while θ increases, the non-zero value of the $2m\dot{r}\dot{\theta}$ term gives rise to a non-zero $\ddot{\theta}$, and hence an acceleration of rotation. If you were sitting next to the particle at the time, you would wonder what caused it to speed up, and you would think that there must have been a force acting upon it.

This is another example of a fictitious force (see section 1.1.3), and is called the Coriolis force. It is used, among other things, to explain why the air rushing in to fill a low pressure area of the atmosphere begins to rotate – thus setting up a 'cyclone'. Some people have attempted to use the equation to explain the direction of rotation of the whirlpool you get above the plughole in a bath.

Put very bluntly – the Coriolis force is the force needed to 'keep' the object going in a true straight line. Of course, a stationary observer would see no force – after all things go in straight lines when there are *no* sideways forces acting on them. The perspective of a rotating observer is not as clear – and this Coriolis force will be felt to be as real as the centrifugal force discussed in section 1.1.3.1.

3.12 *Motion of a rigid body*

When you are dealing with a rigid body, things are simplified in that it can only do two things – move in a line and rotate. If forces $\mathbf{F}_i$ are applied to positions $\mathbf{r}_i$ on a solid object free to move, its motion is completely described by

- a linear acceleration given by $\mathbf{a} = \sum \mathbf{F}_i / M$, where *M* is the total mass of the body, and

- a rotational acceleration given by $\boldsymbol{\alpha} = \sum \mathbf{r}_i' \times \mathbf{F}_i / I$ about a point called the centre of mass, where $\mathbf{r}_i'$ is the position of point *i* relative to the centre of mass and *I* is the moment of inertia of the object about the axis of rotation.[13]

This means, among other things, that the centre of mass itself moves as if it were a point particle of mass *M*. In turn, if a force is applied to the object at the centre of mass, it will cause the body to move with a linear acceleration, without any rotational acceleration at all.

The proof goes as follows. Suppose the object is made up of lots of points $\mathbf{r}_i$ (of mass m_i) fixed together. It follows that Newton's second law states (as in section 1.1.1.2)

[13] This assumes that the angular acceleration is a simple speeding up or slowing down of an existing rotation. If α and ω are not parallel, the situation is more complex.

$$\sum \frac{d(m_i \mathbf{u}_i)}{dt} = \sum \mathbf{F}_i$$
$$\sum \frac{d^2(m_i \mathbf{r}_i)}{dt^2} = \sum \mathbf{F}_i$$
$$\frac{d^2 \sum m_i \mathbf{r}_i}{dt^2} = \sum \mathbf{F}_i$$

Now suppose we define the position **R** such that $M\mathbf{R} = \Sigma\, m_i\, \mathbf{r}_i$, then it follows that

$$M\frac{d^2\mathbf{R}}{dt^2} = \mathbf{F}_{\text{total}}$$

and the point **R** moves as if it were a single point of mass M being acted on by the total force. This position **R** is called the centre of mass.

Given that we already know that **R** does not have any rotational motion, this must be the centre of rotation, and we can use the equation from section 3.10 to show that the rate of change of angular momentum of the object about this point, $d(I\omega)/dt$, is equal to the total torque $\Sigma\, (\mathbf{r}_i - \mathbf{R})\times\mathbf{F}_i$ acting on the body about the point **R**. Given that the masses don't change, we may write

$$\begin{aligned} \frac{d}{dt} m_i \mathbf{u}_i &= \mathbf{F}_i + \sum_j \mathbf{f}_{ij} \\ m_i \mathbf{a}_i &= \mathbf{F}_i + \sum_j \mathbf{f}_{ij} \\ m_i \mathbf{r}_i \times \mathbf{a}_i &= \mathbf{r}_i \times \mathbf{F}_i + \mathbf{r}_i \times \sum_j \mathbf{f}_{ij} \\ \sum_i m_i \mathbf{r}_i \times \mathbf{a}_i &= \sum_i \mathbf{r}_i \times \mathbf{F}_i + \sum_{ij} \mathbf{r}_i \times \mathbf{f}_{ij} \end{aligned}$$

The final term sums to zero since $\mathbf{f}_{ij}+\mathbf{f}_{ji}=\mathbf{0}$, and the internal forces between two particles must either constitute a repulsion, an attraction or the two forces must occur at the same place. In any of these cases $\mathbf{f}_{ij} \times (\mathbf{r}_i - \mathbf{r}j) = \mathbf{0}$.

If we now express the positions $\mathbf{r}_i$ in terms of the centre of mass position **R** and a relative position $\mathbf{r}_i'$, where $\mathbf{r}_i = \mathbf{R} + \mathbf{r}_i'$ (so $\mathbf{a}_i = \mathbf{A}+\mathbf{a}_i'$), then

$$\sum m_i\left(\mathbf{R}+\mathbf{r}_i{}'\right)\times\left(\mathbf{A}+\mathbf{a}_i{}'\right)=\sum\left(\mathbf{R}+\mathbf{r}_i{}'\right)\times\mathbf{F}_i$$

$$\sum m_i\mathbf{R}\times\mathbf{a}_i+\sum m_i\mathbf{r}_i{}'\times\mathbf{A}+\sum m_i\mathbf{r}_i{}'\times\mathbf{a}_i{}'=\sum\mathbf{R}\times\mathbf{F}_i+\sum\mathbf{r}_i{}'\times\mathbf{F}_i$$

$$\sum\mathbf{R}\times m_i\mathbf{a}_i+0+\sum m_i\mathbf{r}_i{}'\times\mathbf{a}_i{}'=\sum\mathbf{R}\times\mathbf{F}_i+\sum\mathbf{r}_i{}'\times\mathbf{F}_i$$

$$\sum m_i\mathbf{r}_i{}'\times\mathbf{a}_i{}'=\sum\mathbf{r}_i{}'\times\mathbf{F}_i$$

since $\Sigma m_i\mathbf{r}_i' = \Sigma m_i(\mathbf{r}_i-\mathbf{R}) = M\mathbf{R} - M\mathbf{R} = \mathbf{0}$. Now, as shown earlier, $d\left(m_i\mathbf{r}_i\times\mathbf{u}_i\right)/dt = m_i\mathbf{u}_i\times\mathbf{u}_i + m_i\mathbf{r}_i\times\mathbf{a}_i = 0 + m_i\mathbf{r}_i\times\mathbf{a}_i$, and so

$$\frac{d}{dt}\sum\mathbf{r}_i{}'\times\mathbf{u}_i=\sum\mathbf{r}_i{}'\times\mathbf{F}_i$$

$$\frac{d}{dt}\mathbf{L}'=\frac{d}{dt}I\boldsymbol{\omega}=\sum\mathbf{r}_i{}'\times\mathbf{F}_i=\mathbf{C}'$$

and so the rate of change of angular momentum about the centre of mass is given by the total moment of the external forces about the centre of mass.

3.13 Questions

1. A car has wheels with radius 30cm. The car travels 42km. By what angle have the wheels rotated during the journey? Make sure that you give your answer in radians and in degrees.

2. Why does the gravitational attraction to the Sun not change the angular momentum of the Earth?

3. Calculate the speed of a satellite orbiting the Earth at a distance of 42 000km from the Earth's centre.

4. A space agency plans to build a spacecraft in the form of a cylinder 50m in radius. The cylinder will be spun so that astronauts inside can walk on the inside of the curved surface as if in a gravitational field of 9.8 N/kg. Calculate the angular velocity needed to achieve this.

5. A television company wants to put a satellite into a 42 000km radius orbit round the Earth. The satellite is launched into a circular low-Earth orbit 200km above the Earth's surface, and a rocket motor then speeds it up. It then coasts until it is in the 42000km orbit with the correct speed. How fast does it need to be going in the low-Earth orbit in order to coast up to the correct position and speed?

6. Estimate the gain in angular velocity when an ice-skater draws her hands in towards her body.

7. One theory of planet formation says that the Earth was once a liquid globule which gradually solidified, and its rotation as a liquid caused it to bulge outwards in the middle – a situation which remains to this day: the equatorial radius of the Earth is about 20km larger than the polar radius. If the theory were correct, what would the rotation rate of the Earth have been just before the crust solidified? Assume that the liquid globule was sufficiently viscous that it was all rotating at the same angular velocity.